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Paired many-to-many disjoint path covers of hypertori

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ABSTRACT

Let n be a positive integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all i. A hypertorus $Q_n^{\mathbf{d}}$ is a simple graph defined on the vertex set $\{(v_1, v_2, \ldots, v_n) : 0 \leq v_i \leq d_i - 1 \text{ for all } i\}$, and has edges between $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if and only if there exists a unique i such that $|u_i - v_i| = 1$ or $d_i - 1$, and for all $j \neq i$, $u_j = v_j$; a two-dimensional hypertorus $Q_2^{\mathbf{d}}$ is simply a torus. In this paper, we prove that if $d_1 \geq 3$ and $d_2 \geq 3$, then $Q_2^{\mathbf{d}}$ is balanced paired 2-to-2 disjoint path coverable if both d_i are even, and is paired 2-to-2 disjoint path coverable otherwise. We also discuss a connection between this result and the popular game Flow Free. Finally, we prove several related results in higher dimensions.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *path of length* n is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$ of vertices and edges such that all vertices v_i, $i = 1, \ldots, n$, are distinct; if instead $v_1 = v_n$, then we call this sequence a *closed path* or *cycle*. A *hamiltonian path* (respectively *hamiltonian cycle*) is a path (cycle) that includes all vertices of G. A graph G is said to be *hamiltonian* if it contains a hamiltonian cycle, and is said to be *hamiltonian connected* if there exists a hamiltonian path between any two distinct vertices in G ([14]; see also e.g. [11]). A bipartite graph G with partite sets V_1 and V_2 is said to be *hamiltonian laceable* if:

- $|V_1| = |V_2|$, and there exists a hamiltonian path between any pair of vertices u in V_1 and v in V_2, or
- $|V_1| = |V_2| + 1$, and there exists a hamiltonian path between any pair of (distinct) vertices u and v in V_1.

Let n be a positive integer, and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ an n-tuple of integers such that $d_i \geq 2$ for all $i = 1, 2, \ldots, n$. Let $L_n^{\mathbf{d}}$ denote an *n-dimensional rectangular lattice*, which is defined to be the graph on the vertex set

$$\{(v_1, v_2, \ldots, v_n) : 0 \leq v_i \leq d_i - 1 \text{ for all } i = 1, 2, \ldots, n\}$$

with edges between $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if and only if there exists a unique i, $1 \leq i \leq n$, such that $|u_i - v_i| = 1$, and for all $j \neq i$, $u_j = v_j$. In 1977, Simmons proved the following theorem.

Theorem 1.1 ([18]). *Every n-dimensional lattice, $n \geq 3$, is hamiltonian laceable.*

Two-dimensional rectangular lattices appear, for example, in the popular game Flow Free [2]. This game is played on a d_1 by d_2 rectangular board, which is equivalent to the rectangular lattice $L_2^{(d_1,d_2)}$. In this game, the player must connect each given pair of dots using a path such that every square is covered and no path crosses itself or any other path. For example, Fig. 1 shows a game board along with one possible solution.

However, it should not be surprising that many boards in two dimensions will not have a solution. For example, in Fig. 2(a), no solution exists and any attempt results in a situation similar to the one pictured in Fig. 2(b).

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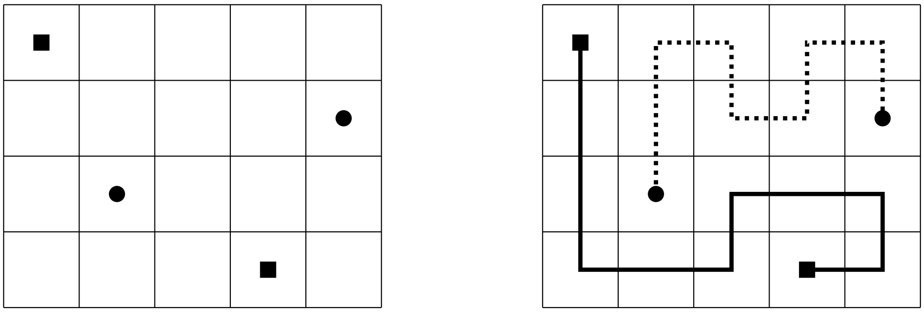

Fig. 1. A Flow Free game on a rectangular lattice along with one possible solution.

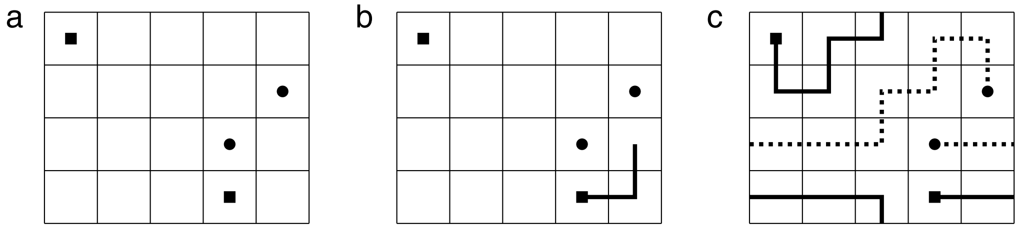

Fig. 2. A game (Fig. 2(a)) which has no solution when played on a rectangular lattice (Fig. 2(b)). However, this game does have a solution when played on a torus (Fig. 2(c)).

This problem occurs because while rectangular lattices are natural objects, they are irregular in the sense that not all vertices are of the same degree. Hence, we will now consider the same game on a torus, meaning that opposite edges are identified and a path can exit one edge of the board and reenter at the corresponding square on the opposite edge. Notice that the game in Fig. 2(a) can now be solved in this environment (see Fig. 2(c)). This encourages the question of how the game can be set-up so that a solution exists; for additional examples of solvable configurations, see [13]. Before stating our theorem that addresses this question, we will need some additional terminology.

Let $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$ be disjoint sets of distinct vertices. In this paper, a *paired disjoint k-path cover* of a graph G is a subgraph of G consisting of paths $P_1, P_2, \dots, P_k$ such that the path P_i has endpoints s_i and t_i, and the vertex sets of the paths partition $V(G)$. We will often abbreviate the term "paired disjoint k-path cover" as "paired k-path cover" or simply "k-path cover". If G has a k-path cover for every choice of S and T, then G is said to be *paired k-to-k disjoint path coverable*.

Let G be a bipartite graph with partite sets V_1 and V_2. Let $|V_1| - |V_2| = \delta$. We say that $S \cup T$ is *balanced* if $|(S \cup T) \cap V_1| - |(S \cup T) \cap V_2| = 2\delta$. Note that the existence of a k-path cover of G for endpoints S and T implies $S \cup T$ is balanced. Hence, G is said to be *balanced paired k-to-k disjoint path coverable* if G has a k-path cover for every choice of S and T such that $S \cup T$ is balanced.

Let $\mathbf{d} = (d_1, d_2, \dots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all i. The graph $Q_n^{\mathbf{d}}$ is a *hypertorus*, or an *n-dimensional torus*, which is defined on the same vertex set as $L_n^{\mathbf{d}}$ and has edges between $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ if and only if there exists a unique i, $1 \leq i \leq n$, such that $|u_i - v_i| = 1$ or $d_i - 1$, and for all $j \neq i$, $u_j = v_j$. A standard torus is simply $Q_2^{(d_1,d_2)}$.

We are now ready to state our main result.

Theorem 3.1. *Let $d_1 \geq 3$ and $d_2 \geq 3$ be integers. If d_1 and d_2 are not both even, then $Q_2^{(d_1,d_2)}$ is paired 2-to-2 disjoint path coverable. Otherwise, $Q_2^{(d_1,d_2)}$ is balanced paired 2-to-2 disjoint path coverable.*

In terms of the game Flow Free, Theorem 3.1 implies that given two pairs of endpoints on a d_1 by d_2 board with opposite sides identified, there exists a solution if and only if either d_1 and d_2 are not both even, or d_1 and d_2 are both even and the four endpoints are balanced, i.e., two endpoints are on white squares and the other two on black squares (when the board is colored like a standard chess board).

We now consider this game in higher dimensions on $Q_n^{\mathbf{d}}$. Our results are as follows. Theorem 2.2 addresses a game scenario in which for each pair, one dot is given and the other may be chosen by the player. Theorem 3.3 returns to the standard Flow Free scenario when played on a torus, but in higher dimensions.

Theorem 2.2. *Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k$ be k distinct vertices of $Q_n^{\mathbf{d}}$. Then there exist k distinct vertices $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k$ of $Q_n^{\mathbf{d}}$, $\mathbf{f}_i \neq \mathbf{e}_j$ if $i \neq j$, such that $Q_n^{\mathbf{d}}$ can be partitioned into vertex-disjoint paths with endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$. Note that if $\mathbf{e}_i = \mathbf{f}_i$, then the path is empty.*

Theorem 3.3. *Let $n \geq 2$ be an integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 3$ for all $i = 1, 2, \ldots, n$.*

1. *Suppose $d_1, d_2, \ldots, d_n$ are not all even. If $Q_n^{\mathbf{d}}$ is paired n-to-n disjoint path coverable for all $\mathbf{d}$ such that $3 \leq d_1, d_2, \ldots, d_n \leq 4n$, then $Q_n^{\mathbf{d}}$ is paired n-to-n disjoint path coverable for all $\mathbf{d}$.*
2. *Suppose $d_1, d_2, \ldots, d_n$ are all even. If $Q_n^{\mathbf{d}}$ is balanced paired n-to-n disjoint path coverable for all $\mathbf{d}$ such that $3 \leq d_1, d_2, \ldots, d_n \leq 4n$, then $Q_n^{\mathbf{d}}$ is balanced paired n-to-n disjoint path coverable for all $\mathbf{d}$.*

Hypertori are interesting graphs to study for many reasons. First, they are regular graphs, i.e. every vertex shares the same degree. In fact, they are vertex transitive graphs, i.e. for any two distinct vertices $\mathbf{u}$ and $\mathbf{v}$ in $Q_n^{\mathbf{d}}$, there is a graph automorphism ϕ on $Q_n^{\mathbf{d}}$ such that $\phi(\mathbf{u}) = \mathbf{v}$. Also, they are generalizations of hypercubes Q_n, studied for example in [4–7,16] and d-ary n-cubes Q_n^d, studied for example in [3,17]. Indeed, when $\mathbf{d} = (2, 2, \ldots, 2)$, $Q_n^{\mathbf{d}} = Q_n$, and when $\mathbf{d} = (d, d, \ldots, d)$, $Q_n^{\mathbf{d}} = Q_n^d$.

The study of hypercubes is strongly motivated by coding theory. While binary codes are objects of fundamental interest, d-ary codes for $d > 2$ are also of great importance. However, these codes exhibit drastically different properties. For example, hypercubes Q_n are regular of degree n, while d-ary n-cubes Q_n^d with $d > 2$ are regular of degree $2n$. Although hypercubes and d-ary hypercubes are natural objects in coding theory, they may also be viewed as geometric objects, in which case there is no reason to limit ourselves to the special case where all dimensions have the same size. Hypertori are the strongest such generalization.

The problem of finding k-path covers is a generalization of the renowned problem of finding hamiltonian paths. In fact, when $k = 1$, a graph is paired 1-to-1 disjoint path coverable (respectively balanced paired 1-to-1 disjoint path coverable) if and only if it is hamiltonian connected (respectively hamiltonian laceable). Furthermore, k-path covers are used in data routing problems to address issues related to communication congestion. Hence, determining whether a graph is paired k-to-k disjoint path coverable has become an interesting topic, and some related results can be found for example in [4,6,8–10,12,15,19].

There are also many variants of the disjoint path cover problem. For example, [17] studies k-disjoint path covers where the paths are disjoint except that they all share a common pair of endpoints. Alternatively, [1] declares a set of vertices T and studies unpaired disjoint path covers where all vertices in T, and possibly others in addition, are used as path endpoints.

2. Fixed-to-floating disjoint path covers of hypertori

In this section, we investigate fixed-to-floating disjoint path covers, which means that one endpoint of each path is predetermined and the other may be chosen as needed to achieve the desired objective. This problem is motivated by its use in the proof of Theorem 3.3. We begin with the following proposition, which is a modest generalization of an analogous result about Q_n^d in [3].

Proposition 2.1. *Let n be a positive integer and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all $i = 1, 2, \ldots, n$. Then the n-dimensional hypertorus $Q_n^{\mathbf{d}}$ is hamiltonian, except when $n = 1$ and $\mathbf{d} = (2)$.*

Proof. We will proceed by induction on n. To simplify our proof, we find it convenient to construct directed cycles; this orientation may of course be removed afterwards.

When $n = 1$ and $d_1 \neq 2$, $Q_1^{(d_1)}$ is the hamiltonian cycle $(0) \to (1) \to \cdots \to (d_1 - 1) \to (0)$. When $n = 2$ and $\mathbf{d} = (2, d_2)$, $Q_2^{\mathbf{d}}$ contains the hamiltonian cycle

$$(0, 0) \to (0, 1) \to \cdots \to (0, d_2 - 1) \to (1, d_2 - 1) \to (1, d_2 - 2) \to \cdots \to (1, 0) \to (0, 0).$$

Assume that $Q_n^{\mathbf{d}}$ has a hamiltonian cycle $\mathbf{v}_0 \to \mathbf{v}_1 \to \cdots \to \mathbf{v}_{D-1} \to \mathbf{v}_0$ for some n, where $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, and $D = d_1 d_2 \cdots d_n$. Let $d_{n+1} \geq 2$ be an integer, and let $\mathbf{d}^* = (d_1, d_2, \ldots, d_{n+1})$. For each vertex $\mathbf{v}$ in $Q_n^{\mathbf{d}}$, let $(\mathbf{v}, j)$ denote the vertex in $Q_{n+1}^{\mathbf{d}^*}$ obtained by appending j to the n-tuple $\mathbf{v}$. It is worth noting that $(\mathbf{v}_0, j) \to (\mathbf{v}_1, j) \to \cdots \to (\mathbf{v}_{D-1}, j) \to (\mathbf{v}_0, j)$ is a cycle (but not a hamiltonian cycle) in $Q_n^{\mathbf{d}}$.

If d_{n+1} is even, then there is a hamiltonian cycle

$$\begin{array}{ccccccccc}
 & (\mathbf{v}_0, 0) & \to & (\mathbf{v}_1, 0) & \to & \cdots & \to & (\mathbf{v}_{D-1}, 0) \\
 & & & & & & & \downarrow \\
 & (\mathbf{v}_0, 1) & \leftarrow & (\mathbf{v}_1, 1) & \leftarrow & \cdots & \leftarrow & (\mathbf{v}_{D-1}, 1) \\
 & \downarrow & & & & & & \\
 & & & & & \ddots & & \\
 & & & & & & & \downarrow \\
(\mathbf{v}_0, 0) \leftarrow & (\mathbf{v}_0, d_{n+1} - 1) & \leftarrow & (\mathbf{v}_1, d_{n+1} - 1) & \leftarrow & \cdots & \leftarrow & (\mathbf{v}_{D-1}, d_{n+1} - 1)
\end{array}$$

in $Q_{n+1}^{\mathbf{d}^*}$.

If d_{n+1} is odd, then there is a hamiltonian cycle

$$\begin{array}{ccccccccc}
(\mathbf{v}_0, 0) & \rightarrow & (\mathbf{v}_1, 0) & \rightarrow & \cdots & \rightarrow & (\mathbf{v}_{D-1}, 0) & & (\mathbf{v}_0, 0) \\
 & & & & & & \downarrow & & \uparrow \\
 & & (\mathbf{v}_1, 1) & \leftarrow & \cdots & \leftarrow & (\mathbf{v}_{D-1}, 1) & & (\mathbf{v}_0, 1) \\
 & & \downarrow & & & & & & \uparrow \\
 & & & & \cdots & & & & \vdots \\
 & & & & & & \vdots & & \vdots \\
 & & & & \cdots & & & & \vdots \\
 & & \downarrow & & & & & & \uparrow \\
 & & (\mathbf{v}_1, d_{n+1}-1) & \rightarrow & \cdots & \rightarrow & (\mathbf{v}_{D-1}, d_{n+1}-1) & \rightarrow & (\mathbf{v}_0, d_{n+1}-1)
\end{array}$$

in $Q_{n+1}^{\mathbf{d}^*}$. $\square$

We now prove the main result of this section.

Theorem 2.2. *Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k$ be k distinct vertices of $Q_n^{\mathbf{d}}$. Then there exist k distinct vertices $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k$ of $Q_n^{\mathbf{d}}$, $\mathbf{f}_i \neq \mathbf{e}_j$ if $i \neq j$, such that $Q_n^{\mathbf{d}}$ can be partitioned into vertex-disjoint paths with endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$. Note that if $\mathbf{e}_i = \mathbf{f}_i$, then the path is empty.*

Proof. If $n = 1$ and $\mathbf{d} = (2)$, then this theorem holds trivially. Otherwise, let $\mathbf{d} = (d_1, d_2, \dots, d_n)$, and let $D = d_1 d_2 \cdots d_n$. By Proposition 2.1, $Q_n^{\mathbf{d}}$ contains a hamiltonian cycle $\mathcal{H} = \mathbf{v}_0 \to \mathbf{v}_1 \to \cdots \to \mathbf{v}_{D-1} \to \mathbf{v}_0$. For all $j = 1, 2, \dots, k$, let i_j be the integer such that $\mathbf{v}_{i_j} = \mathbf{e}_j$. Without loss of generality, assume that $0 = i_1 < i_2 < \cdots < i_k$.

Let $\mathbf{f}_k = \mathbf{v}_{D-1}$, and for all $j = 1, 2, \dots, k-1$, let $\mathbf{f}_j = \mathbf{v}_{i_{j+1}-1}$. Define P_j to be the subpath of $\mathcal{H}$ from $\mathbf{e}_j$ to $\mathbf{f}_j$. In this way, we obtain a vertex-disjoint path-partition of $Q_n^{\mathbf{d}}$ into $P_1, P_2, \dots, P_k$, as desired. $\square$

3. Paired many-to-many disjoint path covers of hypertori

In this section, we begin by proving our main result on paired disjoint path covers of $Q_n^{\mathbf{d}}$ for $n = 2$. We then use a similar argument to prove the induction step for $n \geq 3$.

Theorem 3.1. *Let $d_1 \geq 3$ and $d_2 \geq 3$ be integers. If d_1 and d_2 are not both even, then $Q_2^{(d_1,d_2)}$ is paired 2-to-2 disjoint path coverable. Otherwise, $Q_2^{(d_1,d_2)}$ is balanced paired 2-to-2 disjoint path coverable.*

Proof. We proceed by induction on d_1 and d_2. All base cases, namely $3 \leq d_1 \leq d_2 \leq 8$, have been checked with a computer. In this proof, we will only show the induction on d_2, as induction on d_1 is analogous. Fix $d_2 \geq 7$, and suppose that $Q_2^{(d_1,d_2)}$ is (balanced) paired 2-to-2 disjoint path coverable.

Consider $Q_2^{(d_1,d_2+2)}$ with a given choice of endpoints $\{\mathbf{s}_1, \mathbf{s}_2\}$ and $\{\mathbf{t}_1, \mathbf{t}_2\}$ (with the additional condition that $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_1, \mathbf{t}_2\}$ is balanced when d_1 and d_2 are both even). Let column j of $Q_2^{(d_1,d_2+2)}$ denote the set of vertices $\{(i, j) : i = 0, 1, \dots, d_1 - 1\}$. As $d_2 + 2 \geq 9$, there are two consecutive columns of $Q_2^{(d_1,d_2+2)}$ that contain none of the four endpoints. Without loss of generality, let them be columns d_2 and $d_2 + 1$ of $Q_2^{(d_1,d_2+2)}$.

Let R denote the subgraph of $Q_2^{(d_1,d_2+2)}$ induced by columns $0, 1, \dots, d_2 - 1$. Let Q be the 2-dimensional torus obtained from R by connecting $(i, 0)$ and $(i, d_2 - 1)$ for all $i = 0, 1, \dots, d_1 - 1$. Note that Q is isomorphic to $Q_2^{(d_1,d_2)}$. By the induction hypothesis, there exists a paired 2-path cover $\mathcal{C}$ of Q with endpoints $\{\mathbf{s}_1, \mathbf{s}_2\}$ and $\{\mathbf{t}_1, \mathbf{t}_2\}$. Define

$$X = \big\{(x_i, d_2 - 1) : \{(x_i, d_2 - 1), (x_i, 0)\} \in E(\mathcal{C})\big\},$$

the set of vertices in column $d_2 - 1$ of Q from which the 2-path cover $\mathcal{C}$ has an edge to a vertex in column 0 of Q. Here, $E(\mathcal{C})$ denotes the edge set of $\mathcal{C}$.

If X is empty, then the 2-path cover $\mathcal{C}$ contains an edge in Q between two vertices $(y, d_2 - 1)$ and $(y + 1, d_2 - 1)$, where $y < d_1 - 1$. We now construct a 2-path cover of $Q_2^{(d_1,d_2+2)}$. Embed $\mathcal{C}$ into $Q_2^{(d_1,d_2+2)}$, and denote the image as $\mathcal{C}'$. The edge set of the 2-path cover of $Q_2^{(d_1,d_2+2)}$ will be

$$\begin{aligned}
&\big(E(\mathcal{C}') \setminus \big\{\{(y, d_2 - 1), (y + 1, d_2 - 1)\}\big\}\big) \\
&\quad \cup \big\{\{(y, d_2 - 1), (y, d_2)\}, \{(y + 1, d_2 - 1), (y + 1, d_2)\}, \{(0, d_2), (0, d_2 + 1)\}, \\
&\{(d_1 - 1, d_2), (d_1 - 1, d_2 + 1)\}\big\} \cup \big\{\{(i, d_2), (i + 1, d_2)\} : i = 0, 1, \dots, y - 1, y + 1, y + 2, \dots, d_1 - 2\big\} \\
&\quad \cup \big\{\{(i, d_2 + 1), (i + 1, d_2 + 1)\} : i = 0, 1, \dots, d_1 - 2\big\}.
\end{aligned}$$

If X is nonempty, then let $k = |X|$. Let $\mathbf{e}_i = (x_i, d_2)$ for all $i = 1, 2, \ldots, k$, and let $\widehat{Q}$ and $\widehat{Q}'$ denote the subgraphs of $Q_2^{(d_1,d_2+2)}$ induced by columns d_2 and $d_2 + 1$ respectively. Note that $\widehat{Q}$ is isomorphic to $Q_1^{(d_1)}$. Furthermore, note that $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ are distinct vertices in $\widehat{Q}$. By Theorem 2.2, there exist k distinct vertices $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k$ of $\widehat{Q}$, $\mathbf{f}_i \neq \mathbf{e}_j$ if $i \neq j$, such that $\widehat{Q}$ can be partitioned into vertex-disjoint paths P_i with endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$. Duplicate $\widehat{Q}$, including all the endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$ and paths P_i, into $\widehat{Q}'$. Denote the images of $\mathbf{e}_i$, $\mathbf{f}_i$, and P_i in $\widehat{Q}'$ as $\mathbf{e}_i'$, $\mathbf{f}_i'$, and P_i' respectively.

We now construct a 2-path cover of $Q_2^{(d_1,d_2+2)}$ as follows. Embed

$$E(\mathcal{C}) \setminus \left\{\{(x_i, d_2 - 1), (x_i, 0)\} : i = 1, 2, \ldots, k\right\}$$

into $Q_2^{(d_1,d_2+2)}$, and denote the image as $\mathcal{C}''$. The 2-path cover of $Q_2^{(d_1,d_2+2)}$ will then consist of all edges from $\mathcal{C}''$ and

$$\left\{\{(x_i, d_2 - 1), \mathbf{e}_i\}, \{\mathbf{e}_i', (x_i, 0)\}, \{\mathbf{f}_i, \mathbf{f}_i'\} : i = 1, 2, \ldots, k\right\} \cup \bigcup_{i=1}^{k} E(P_i) \cup \bigcup_{i=1}^{k} E(P_i'). \quad \square$$

We now state a conjecture that generalizes the above theorem to all $n \geq 2$.

Conjecture 3.2. *Let $n \geq 2$ be an integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 3$ for all $i = 1, 2, \ldots, n$. If $d_1, d_2, \ldots, d_n$ are not all even, then $Q_n^{\mathbf{d}}$ is paired n-to-n disjoint path coverable. Otherwise, $Q_n^{\mathbf{d}}$ is balanced paired n-to-n disjoint path coverable.*

This conjecture is the strongest possible in the sense that $Q_n^{\mathbf{d}}$ cannot be $(n + 1)$-to-$(n + 1)$ disjoint path coverable. This is because if we pick the endpoints $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ to be neighbors of $\mathbf{s}_{n+1}$, then there is obviously no path that joins $\mathbf{s}_{n+1}$ with $\mathbf{t}_{n+1}$ since every vertex in $Q_n^{\mathbf{d}}$ has degree $2n$.

Theorem 3.1 proves Conjecture 3.2 in the case $n = 2$. For $n \geq 3$, we can prove an inductive step analogous to that in the proof of Theorem 3.1.

Theorem 3.3. *Let $n \geq 2$ be an integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 3$ for all $i = 1, 2, \ldots, n$.*

1. *Suppose $d_1, d_2, \ldots, d_n$ are not all even. If $Q_n^{\mathbf{d}}$ is paired n-to-n disjoint path coverable for all $\mathbf{d}$ such that $3 \leq d_1, d_2, \ldots, d_n \leq 4n$, then $Q_n^{\mathbf{d}}$ is paired n-to-n disjoint path coverable for all $\mathbf{d}$.*
2. *Suppose $d_1, d_2, \ldots, d_n$ are all even. If $Q_n^{\mathbf{d}}$ is balanced paired n-to-n disjoint path coverable for all $\mathbf{d}$ such that $3 \leq d_1, d_2, \ldots, d_n \leq 4n$, then $Q_n^{\mathbf{d}}$ is balanced paired n-to-n disjoint path coverable for all $\mathbf{d}$.*

Proof. The case for $n = 2$ was finished in Theorem 3.1. We now assume $n \geq 3$, and proceed by induction on $d_1, d_2, \ldots, d_n$. All base cases, namely $3 \leq d_1, d_2, \ldots, d_n \leq 4n$, hold by assumption. In this proof, we will only show the induction on d_n, as the induction steps on $d_1, d_2, \ldots, d_{n-1}$ are analogous. Fix $d_n \geq 4n - 1$, and suppose that $Q_n^{\mathbf{d}}$ is (balanced) paired n-to-n disjoint path coverable.

Let $\mathbf{d}^* = (d_1, d_2, \ldots, d_{n-1}, d_n + 2)$. Consider $Q_n^{\mathbf{d}^*}$ with a given choice of endpoints $S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ and $T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n\}$ (with the additional condition that $S \cup T$ is balanced when $d_1, d_2, \ldots, d_n$ are all even). For every $j = 0, 1, \ldots, d_n + 1$, let layer j of $Q_n^{\mathbf{d}^*}$ denote the set of vertices

$$\{(i_1, i_2, \ldots, i_{n-1}, j) : 0 \leq i_\ell \leq d_\ell - 1 \text{ for all } \ell = 1, \ldots, n - 1\}.$$

As $d_n + 2 \geq 4n + 1$, there are two consecutive layers of $Q_n^{\mathbf{d}^*}$ that contain none of the endpoints in $S \cup T$. Without loss of generality, let them be layers d_n and $d_n + 1$ of $Q_n^{\mathbf{d}^*}$.

Let R denote the subgraph of $Q_n^{\mathbf{d}^*}$ induced by layers $0, 1, \ldots, d_n - 1$. Let Q be the n-dimensional hypertorus obtained from R by connecting the vertex $(i_1, i_2, \ldots, i_{n-1}, 0)$ to the vertex $(i_1, i_2, \ldots, i_{n-1}, d_n - 1)$ for each $0 \leq i_\ell \leq d_\ell - 1$, $\ell = 1, \ldots, n - 1$. Note that Q is isomorphic to $Q_n^{\mathbf{d}}$. By the induction hypothesis, there exists a paired n-path cover $\mathcal{C}$ of Q with endpoints S and T. In the following, we will use $(\mathbf{x}, j)$ to denote the vertex $(x_1, x_2, \ldots, x_{n-1}, j)$. Define

$$X = \left\{(\mathbf{x}_i, d_n - 1) : \{(\mathbf{x}_i, d_n - 1), (\mathbf{x}_i, 0)\} \in E(\mathcal{C})\right\},$$

the set of vertices in layer $d_n - 1$ of Q from which the n-path cover $\mathcal{C}$ has an edge to a vertex in layer 0 of Q. Here, $E(\mathcal{C})$ denotes the edge set of $\mathcal{C}$.

If X is empty, then the n-path cover $\mathcal{C}$ contains an edge in Q between two vertices

$$\mathbf{y}_1 = (y_1, y_2, \ldots, y_\ell, \ldots, y_{n-1}, d_n - 1)$$

and

$$\mathbf{y}_2 = (y_1, y_2, \ldots, y_\ell + 1, \ldots, y_{n-1}, d_n - 1).$$

Let L denote the n-dimensional rectangular lattice on the vertices in layers d_n and d_{n+1} of $Q_n^{\mathbf{d}^*}$. By Theorem 1.1, there exists a 1-path cover $\mathcal{D}$ of L with endpoints

$$\mathbf{y}_1' = (y_1, y_2, \ldots, y_\ell, \ldots, y_{n-1}, d_n) \quad \text{and} \quad \mathbf{y}_2' = (y_1, y_2, \ldots, y_\ell + 1, \ldots, y_{n-1}, d_n).$$

We now construct an n-path cover of $Q_n^{\mathbf{d}^*}$. Embed $\mathcal{C}$ into $Q_n^{\mathbf{d}^*}$, and denote the image as $\mathcal{C}'$. The edge set of the n-path cover of $Q_n^{\mathbf{d}^*}$ will be

$$\left(E(\mathcal{C}') \setminus \left\{\{\mathbf{y}_1, \mathbf{y}_2\}\right\}\right) \cup \left\{\{\mathbf{y}_1, \mathbf{y}_1'\}, \{\mathbf{y}_2, \mathbf{y}_2'\}\right\} \cup E(\mathcal{D}).$$

If X is nonempty, then let $k = |X|$. Let $\mathbf{e}_i = (\mathbf{x}_i, d_n)$ for all $i = 1, 2, \ldots, k$, and let $\widehat{Q}$ and $\widehat{Q}'$ denote the subgraphs of $Q_n^{\mathbf{d}^*}$ induced by layers d_n and $d_n + 1$ respectively. Note that $\widehat{Q}$ is isomorphic to $Q_{n-1}^{(d_1, d_2, \ldots, d_{n-1})}$. Furthermore, note that $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ are distinct vertices in $\widehat{Q}$. By Theorem 2.2, there exist k distinct vertices $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k$ of $\widehat{Q}$, $\mathbf{f}_i \neq \mathbf{e}_j$ if $i \neq j$, such that $\widehat{Q}$ can be partitioned into vertex-disjoint paths P_i with endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$. Duplicate $\widehat{Q}$, including all the endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$ and paths P_i, into $\widehat{Q}'$. Denote the images of $\mathbf{e}_i$, $\mathbf{f}_i$, and P_i in $\widehat{Q}'$ as $\mathbf{e}_i'$, $\mathbf{f}_i'$, and P_i' respectively.

We now construct a paired n-path cover of $Q_n^{\mathbf{d}^*}$ as follows. Embed

$$E(\mathcal{C}) \setminus \left\{\{(\mathbf{x}_i, d_n - 1), (\mathbf{x}_i, 0)\} : i = 1, 2, \ldots, k\right\}$$

into $Q_n^{\mathbf{d}^*}$, and denote the image as $\mathcal{C}''$. The n-path cover of $Q_n^{\mathbf{d}^*}$ will then consist of all edges from $\mathcal{C}''$ and

$$\left\{\{(\mathbf{x}_i, d_n - 1), \mathbf{e}_i\}, \{\mathbf{e}_i', (\mathbf{x}_i, 0)\}, \{\mathbf{f}_i, \mathbf{f}_i'\} : i = 1, 2, \ldots, k\right\} \cup \bigcup_{i=1}^{k} E(P_i) \cup \bigcup_{i=1}^{k} E(P_i'). \quad \square$$

4. Concluding remarks and further directions

There are many studies about disjoint path covers and hypercubes. Their best results usually obtain at most $\lfloor n/2 \rfloor$-path covers for Q_n because Q_n is regular of degree n. For example, see [10]. In contrast, the hypertori explored in this paper are regular of degree $2n$. This illustrates a fundamental difference between these two families of objects, and motivates us to extend results on hypercubes to hypertori.

There are several possible directions in which this work could be extended. One is to prove Conjecture 3.2 by completing the base case. Another is to study the impact of faulty elements on these problems.

Alternatively, one could generalize the results of this paper to relax the restrictions on d_i. Consider $Q_n^{\mathbf{d}}$, where $d_i \geq 2$ for all i, and let h denote the cardinality of the set $\{d_i : d_i = 2\}$. Note that $Q_n^{\mathbf{d}}$ is regular of degree $2n - h$. We end with the following conjecture.

Conjecture 4.1. *Let $n \geq 2$ be an integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all $i = 1, 2, \ldots, n$. Define $h = |\{d_i : d_i = 2, i = 1, 2, \ldots, n\}|$. If $d_1, d_2, \ldots, d_n$ are not all even, then $Q_n^{\mathbf{d}}$ is paired $\left(n - \lceil \frac{h}{2} \rceil\right)$-to-$\left(n - \lceil \frac{h}{2} \rceil\right)$ disjoint path coverable. Otherwise, $Q_n^{\mathbf{d}}$ is balanced paired $\left(n - \lceil \frac{h}{2} \rceil\right)$-to-$\left(n - \lceil \frac{h}{2} \rceil\right)$ disjoint path coverable.*

Note that when $h = n$, i.e. $Q_n^{\mathbf{d}}$ is a hypercube, this conjecture states that Q_n is balanced paired $\lfloor n/2 \rfloor$-disjoint path coverable, which is consistent with the results mentioned above.

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