MAT 260 LINEAR ALGEBRA
LECTURE 20

WING HONG TONY WONG

1.3 — Matrices and matrix operations

There are two other ways to view matrix multiplication.

(1) If
$$B = (\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_c),$$
where $\mathbf{b}_i$ represents the i-th column of B as a column vector, then
$$AB = (A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_c).$$

(2) If
$$A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_r \end{pmatrix},$$
where $\mathbf{a}_i$ represents the i-th row of A as a row vector, then
$$AB = \begin{pmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_r B \end{pmatrix}.$$

Also, if
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{pmatrix}$$
is a column vector of length c, and $A = (\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n)$ is of dimensions $r \times n$, where $\mathbf{a}_i$ represents the i-th column of A, then $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$. We say that it is a **linear combination** of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ with coefficients $x_1, x_2, \ldots, x_n$.

Here are two more operations of matrices:

- **Transpose** of A, denoted by $A^\top$, flips rows and columns. The ij-th entry of $A^\top$ is a_{ji}. The i-th row of A is the i-th column of $A^\top$, and the j-th column of A is the j-th row of $A^\top$. If A is of dimensions $r \times c$, then $A^\top$ is of dimensions $c \times r$. It is not hard to see that for any scalars k and ℓ, if A and B are of the same size, then $(kA + \ell B)^\top = kA^\top + \ell B^\top$.

Date: Friday, March 6, 2020.

- If A is a square matrix, then the **trace** of A, denoted by $\operatorname{tr}(A)$, is defined as the sum of the diagonal entries of A, i.e. $\operatorname{tr}(A) = \sum a_{ii}$. Again, it is not hard to see that for any scalars k and ℓ, $\operatorname{tr}(kA + \ell B) = k\operatorname{tr}(A) + \ell\operatorname{tr}(B)$.

Theorem 1. *If AB is defined, then $(AB)^\top = B^\top A^\top$.*

Theorem 2. *Let A and B be both square matrices of order n. Then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.*

Warning: In general, $\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$.

Example 3. Here are some subspaces of M_{nn}, the set of all $n \times n$ **square** matrices.

- Sets of $n \times n$ **upper-triangular** matrices: $\mathcal{U}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i > j\}$.
- Sets of $n \times n$ **lower-triangular** matrices: $\mathcal{L}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i < j\}$.
- Sets of $n \times n$ **diagonal** matrices: $\mathcal{D}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i \neq j\}$.
- Sets of $n \times n$ **symmetric** matrices: $\{A \in M_{nn} : a_{ij} = a_{ji} \text{ for all } i, j\}$, or $\{A \in M_{nn} : A^\top = A\}$.
- Sets of $n \times n$ **skew-symmetric** matrices: $\{A \in M_{nn} : a_{ij} = -a_{ji} \text{ for all } i, j\}$, or $\{A \in M_{nn} : A^\top = -A\}$.