MAT 260 LINEAR ALGEBRA
LECTURE 18

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1.2 — More on Gaussian Elimination

From the last section, a recipe for solving a system of linear equations is the following.

(1) Set up an augmented matrix.
(2) Perform a sequence of elementary row operations on the augmented matrix:
   - The forward phase produces a row echelon form of the augmented matrix.
   - The backward phase produces the reduced row echelon form (rref) of the augmented matrix.
(3) Interpret the solutions from the rref of the augmented matrix:
   - If there is a leading 1 in the last **column**, then there is no solution.
   - If there is a leading 1 in every **column** except the last, then there is a unique solution.
   - If there is no leading 1 in the last **column** as well as some other **column**, then there are infinitely many solutions. You will need to use free parameters to express the solutions.

A system of linear equations is **homogeneous** if all constant terms are 0s, i.e., all entries in the last column of the augmented matrix are 0s. In a homogeneous system of linear equations, all variables being 0s is always a solution, and we call it the **trivial solution**. Any other solution (if it exists) is called a **nontrivial solution**. Note that a homogeneous system of linear equations always has a unique solution or infinitely many solutions.

Warning: Do NOT abuse the term "trivial solution." It is reserved to refer to the SOLUTION of all 0s for a HOMOGENEOUS system of linear equations. Similarly, the term "nontrivial solution" only refers to a solution that is not all 0s for a HOMOGENEOUS system of linear equations.

Theorem 1. *In a HOMOGENEOUS system of linear equations, if there are more variables than equations, then it has infinitely many solutions.*

Proof. Let the augmented matrix of this homogeneous system of linear equations be

$$\left(\begin{array}{c|c} \quad A \quad & \mathbf{0} \end{array}\right),$$

where A is a matrix of **size** $m \times n$ (read as **m-by-n**, meaning that there are m rows and n columns in A), and the **bold** number $\mathbf{0}$ denotes a column of 0s.

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Since there are more variables than equations, we have $m < n$, i.e., the matrix A has more columns than rows. When the augmented matrix $(A|\mathbf{0})$ is reduced to its reduced row echelon form $(\operatorname{rref} A|\mathbf{0})$, there are at most m leading 1s (at most one per row) in $\operatorname{rref} A$. As a result, there exists a column of $\operatorname{rref} A$ that does not have a leading 1. Hence, there are infinitely many solutions to this homogeneous system of linear equations.

$\square$

Question: Where did we need use the condition that this system is homogeneous?

The next theorem shows that the solution set of a HOMOGENEOUS system of linear equations forms a vector space. Before this theorem, however, we first introduce several matrix operations: equality, addition, subtraction, scalar multiplication, and multiplication.

- $A = B$.
  The two matrices A and B must share the **same sizes** and the **same corresponding entries**. For instance, the following matrices are not equal to each other.
$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 0 & 3 \end{array}\right) \neq \left(\begin{array}{cc|c} 1 & 3 & -2 \\ -1 & 0 & 3 \end{array}\right),$$
$$\begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$
- $A + B$ and $A - B$.
  The two matrices A and B must share the same size, and addition and subtraction are performed **entrywise**. For example,
$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 0 & 3 \end{array}\right) + \left(\begin{array}{cc|c} 0 & -2 & 3 \\ 2 & 4 & -1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 4 & 2 \end{array}\right),$$
$$\begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -1 \\ -3 & -4 & 4 \end{pmatrix}.$$
- kA, where $k \in \mathbb{R}$.
  Scalar multiplication is also performed **entrywise**. For example,
$$3\left(\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 0 & 3 \end{array}\right) = \left(\begin{array}{cc|c} 3 & 9 & 6 \\ -3 & 0 & 9 \end{array}\right),$$
$$\frac{1}{6}\begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}.$$
- AB.
  The two matrices A and B must have **compatible sizes**, i.e., the size of A is $m \times k$, and the size of B is $k \times n$. The product AB has size $m \times n$, and the ij-th entry is the **dot product** between the i-th row of A and the j-th column of B, i.e.,
$$\sum_{\ell=1}^{k} a_{i\ell} b_{\ell j}.$$

For example,

$$\begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} = 1 \cdot (-5) + 3 \cdot (-1) + 2 \cdot 4 = 0,$$

$$\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -6 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 3 \cdot (-2) & 1 \cdot (-6) + 3 \cdot 5 \\ (-1) \cdot 4 + 2 \cdot (-2) & (-1) \cdot (-6) + 2 \cdot 5 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ -8 & 16 \end{pmatrix}.$$

Important: Matrix multiplication is

- **NOT commutative**.
- **associative**.
- **distributive**.

Now, we can state and prove our next theorem.

Theorem 2. *Let A be an $m \times n$ matrix. The solution set to the matrix equation $A\mathbf{x} = \mathbf{0}$, where*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

denotes a column of variables $x_1, x_2, \ldots, x_n$, forms a vector space.

Question: What is the size of this column $\mathbf{0}$?

Proof. The solution set can be written as

$$W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Here, $\mathbf{x}$ represents a column of n real numbers, and $\mathbb{R}^n$ denotes the set of all columns of n real numbers. We want to show that W is a subspace of $\mathbb{R}^n$.

For all $\mathbf{x}, \mathbf{y} \in W$,

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Therefore, $\mathbf{x} + \mathbf{y} \in W$, and Axiom (1) holds for W.

For all $k \in \mathbb{R}$ and $\mathbf{x} \in W$,

$$A(k\mathbf{x}) = k(A\mathbf{x}) = k\mathbf{0} = \mathbf{0}.$$

Therefore, $k\mathbf{x} \in W$, and Axiom (6) holds for W.

By Theorem 3 of Lecture note 9, W is a subspace of $\mathbb{R}^n$.

□