MAT 260 LINEAR ALGEBRA
LECTURE 40

WING HONG TONY WONG

4.4 — Basis and coordinates

Let V be a vector space.

Definition 1. $S \subseteq V$ is a **basis** of V if
- S is linearly independent, and
- $\mathrm{span}(S) = V$.

Note that a basis of a vector space is not unique in general.

Definition 2. V is **finite-dimensional** if V has a basis of finite size. Otherwise, V is **infinite-dimensional**.

Example 3. In $\mathbb{R}^n$, the set of standard unit vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ forms a basis. This is called the standard basis of $\mathbb{R}^n$. In $\mathbb{R}^3$, the set $\{(1, 1, 1), (1, 2, 4), (1, 3, 9)\}$ also forms a basis.

Example 4. In P_n, the set of standard unit vectors, $\{1, x, x^2, \ldots, x^n\}$ forms a basis. This is called the standard basis of P_n. In P_2, the set $\{1, 1 + x, 1 + x + x^2\}$ also forms a basis.

Example 5. In $\mathbb{R}^\infty$, the set of standard unit vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ does not form a basis. However, in P_∞, the set of standard unit vectors $\{1, x, x^2, \ldots\}$ forms a basis. In these two vector spaces, it is easy to see they are both infinite-dimensional.

Example 6. In $M_{mn}(\mathbb{R})$, let A_{ij} the matrix with 1 at the ij-th entry and 0 everywhere else. Then the set of all A_{ij} forms a basis of $M_{mn}(\mathbb{R})$, and it is called the standard basis of $M_{mn}(\mathbb{R})$.

Theorem 7. *If S is a basis of V, then every nonzero vector $\mathbf{v} \in V$ can be expressed uniquely in the form $\mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n$ for some $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S$.*

Let V be a finite-dimensional vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V. By Theorem 7, there is a map $V \to \mathbb{R}^n$ defined by

$$\mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n \mapsto (\mathbf{v})_S = (a_1, a_2, \ldots, a_n).$$

$(\mathbf{v})_S$ is called the **coordinate vector of v relative to** S.

Example 8. In P_2, if $S = \{1, x, x^2\}$, then $(1 - x^2)_S = (1, 0, -1)$. If we take $S' = \{1, 1 + x, 1 + x + x^2\}$ instead, then $(1 - x^2)_{S'} = (1, 1, -1)$.

Example 9. In $\mathbb{R}^3$, if $S = \{(1, 1, 1), (1, 0, 1), (1, 0, -1)\}$, then $(3, 2, 1)_S = (2, 0, 1)$.