

**MAT 260 LINEAR ALGEBRA**  
**LECTURE 36**

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**4.3 — Linear independence**

Let  $V$  be a vector space, and let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ .

**Definition 1.**  $\mathbf{v}_n$  is **linearly dependent** on  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}$  if there exists  $k_1, k_2, \dots, k_{n-1} \in \mathbb{R}$  such that

$$\mathbf{v}_n = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_{n-1}\mathbf{v}_{n-1}.$$

Otherwise,  $\mathbf{v}_n$  is **linearly independent** of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}$ .

**Definition 2.**  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are **linearly dependent** if there exists  $k_1, k_2, \dots, k_n \in \mathbb{R}$ , not all  $k_i$ 's are zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = \mathbf{id}.$$

Otherwise,  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are **linearly independent**.

**Definition 3.**  $S \subseteq V$  is **linearly dependent** if there exists  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in S$  and  $k_1, k_2, \dots, k_n \in \mathbb{R}$ , not all  $k_i$ 's are zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = \mathbf{id}.$$

Otherwise,  $S$  is **linearly independent**. If  $S = \emptyset$ , then  $S$  is also defined to be **linearly independent**.

Recall that a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n.$$

A **trivial linear combination** of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_n,$$

which is obviously equal to  $\mathbf{id}$ . Hence,  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly dependent if there is a **nontrivial linear combination** of them to form  $\mathbf{id}$ .

**Theorem 4.** Let  $S$  be a subset of vectors in a vector space  $V$ . Then

- (a) if  $\mathbf{id} \in S$ ,  $S$  is linearly dependent;
- (b) if  $S$  contains only one vector, then  $S$  is linearly independent if and only if  $\mathbf{id} \notin S$ ;
- (c) if  $S$  contains exactly two vectors, then  $S$  is linearly independent if and only if neither vector is a scalar multiple of the other.

**Theorem 5.** Let  $S$  be a subset of two or more vectors in a vector space  $V$ . Then  $S$  is linearly independent if and only if no vector in  $S$  is a linear combination of the others in  $S$ .

If we want to prove that a set  $S$  is linearly independent, we often use proof by contrapositive or proof by contradiction. This is because linear independence is a “for all” statement:

All nontrivial linear combinations of vectors in  $S$  are not equal to  $\mathbf{id}$ .

Hence, with proof by contrapositive or proof by contradiction, we can start with “there exists” assumption:

There exists a nontrivial combination of vectors in  $S$  that equals  $\mathbf{id}$ .

**Theorem 6.** Let  $V = \mathbb{R}^n$ , and let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ . If  $m > n$ , then  $S$  is linearly dependent.

**Example 7.** In  $\mathbb{R}^\infty$ , let  $\mathbf{e}_1 = (1, 0, 0, \dots)$ ,  $\mathbf{e}_2 = (0, 1, 0, \dots)$ ,  $\dots$ . Then  $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots\}$  is linearly independent.

**Example 8.** In  $V = \mathbb{R}^4$ , let  $S = \{(1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12)\}$ ,  $S' = \{(1, 1, 1, 1), (1, 2, 4, 8), (1, 3, 9, 27), (1, 4, 16, 64)\}$ , and  $S'' = \{(1, 1, 1, 1), (1, 2, 4, 8), (1, 3, 9, 27), (1, 4, 16, 64), (1, 5, 25, 125)\}$ . Then  $S$  and  $S''$  are linearly dependent, while  $S'$  is linearly independent.

**Example 9.** In  $V = P_\infty$ ,  $S = \{1, x, x^2, \dots\}$  is linearly independent, while  $S' = \{1 - x, 5 + 3x - 2x^2, 1 + 3x - x^2\}$  is linearly dependent.

**Theorem 10.** If  $f_1, f_2, \dots, f_n \in F(-\infty, \infty)$  have  $n - 1$  derivatives, then  $\{f_1, f_2, \dots, f_n\}$  are linearly independent if (but NOT only if) the Wronskian of these functions is not identically zero, where the Wronskian is

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}(x) & f^{(n-1)}(x) & \cdots & f^{(n-1)}(x) \end{vmatrix}.$$

**Example 11.** In  $V = F(-\infty, \infty)$ ,  $S = \{2, \sin x\}$  is linearly independent, while  $S' = \{2, \sin^2 x, \cos^2 x\}$  is linearly dependent.