

**MAT 260 LINEAR ALGEBRA  
LECTURE 28**

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**1.6 — More on linear systems and invertible matrices**

**Theorem 1** (1.6.23). *Let  $A\mathbf{x} = \mathbf{b}$  be a consistent system of linear equations, and let  $\mathbf{x}_1$  be any fixed solution. Show that  $\mathbf{x}$  is a solution to  $A\mathbf{x} = \mathbf{b}$  if and only if  $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$  for some solution  $\mathbf{x}_0$  to the equation  $A\mathbf{x}_0 = \mathbf{0}$ .*

If we are solving multiple equations  $A\mathbf{x} = \mathbf{b}_1, A\mathbf{x} = \mathbf{b}_2, \dots, A\mathbf{x} = \mathbf{b}_k$  at the same time, then we can row reduce  $(A|\mathbf{b}_1|\mathbf{b}_2|\dots|\mathbf{b}_k)$ .

**Example 2.** Solve the systems

$$\begin{array}{rcl} x_1 + 2x_2 + 3x_3 = 4 & & x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 5x_2 + 3x_3 = 5 & \text{and} & 2x_1 + 5x_2 + 3x_3 = 6 \\ x_1 + 8x_3 = 9 & & x_1 + 8x_3 = -6 \end{array} .$$

**Theorem 3.** *If  $AB$  is an invertible matrix, and  $A$  and  $B$  are square matrices, then both  $A$  and  $B$  are invertible.*

**Example 4.** What are the conditions on  $b_1, b_2, b_3$  such that

$$\begin{array}{l} x_1 + x_2 + 2x_3 = b_1 \\ x_1 + x_3 = b_2 \\ 2x_1 + x_2 + 3x_3 = b_3 \end{array}$$

is consistent? How about

$$\begin{array}{l} x_1 + x_2 + 2x_3 = b_1 \\ x_1 + x_3 = b_2 ? \\ x_2 + 3x_3 = b_3 \end{array}$$

**1.7 — Diagonal, triangular, and symmetric matrices**

**Example 5.** Once again, recall the symbols for some subspaces of  $M_{nn}$ , the set of all  $n \times n$  square matrices.

- Sets of  $n \times n$  **upper-triangular** matrices:  $\mathcal{U}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i > j\}$ .
- Sets of  $n \times n$  **lower-triangular** matrices:  $\mathcal{L}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i < j\}$ .
- Sets of  $n \times n$  **diagonal** matrices:  $\mathcal{D}_{nn} = \{A \in M_{nn} : a_{ij} = 0 \text{ for all } i \neq j\}$ .

- Sets of  $n \times n$  **symmetric** matrices:  $\{A \in M_{nn} : a_{ij} = a_{ji} \text{ for all } i, j\}$ , or  $\{A \in M_{nn} : A^\top = A\}$ .
- Sets of  $n \times n$  **skew-symmetric** matrices:  $\{A \in M_{nn} : a_{ij} = -a_{ji} \text{ for all } i, j\}$ , or  $\{A \in M_{nn} : A^\top = -A\}$ .

Here are some properties of these matrices:

- If  $A \in \mathcal{U}_{nn}$  (respectively  $A \in \mathcal{L}_{nn}$  or  $A \in \mathcal{D}_{nn}$ ), then  $A$  is invertible if and only if all diagonal entries are nonzero. Besides, if  $A$  is invertible, then  $A^{-1} \in \mathcal{U}_{nn}$  (respectively  $A^{-1} \in \mathcal{L}_{nn}$  or  $A^{-1} \in \mathcal{D}_{nn}$ ), and all the diagonal entries in  $A^{-1}$  are reciprocals of those in  $A$ .
- If  $A, B \in \mathcal{U}_{nn}$  (respectively  $A, B \in \mathcal{L}_{nn}$  or  $A, B \in \mathcal{D}_{nn}$ ), then  $AB \in \mathcal{U}_{nn}$  (respectively  $AB \in \mathcal{L}_{nn}$  or  $AB \in \mathcal{D}_{nn}$ ). Also, the  $ii$ -th entry of  $AB$  is  $a_{ii}b_{ii}$ . Consequently, the  $ii$ -th entry of  $A^n$  is  $a_{ii}^n$ .
- If  $A \in \mathcal{U}_{nn}$  (respectively  $A \in \mathcal{L}_{nn}$ ), then  $A^\top \in \mathcal{L}_{nn}$  (respectively  $A^\top \in \mathcal{U}_{nn}$ ).
- If  $A \in \mathcal{D}_{nn}$ , then the  $i$ -th row of  $AB$  is the scalar product between  $a_{ii}$  and the  $i$ -th row of  $B$ , and the  $j$ -th column of  $BA$  is the scalar product between  $a_{ii}$  and the  $j$ -th column of  $B$ .
- $\mathcal{U}_{nn} \cap \mathcal{L}_{nn} = \mathcal{D}_{nn}$ .
- If  $A$  and  $B$  are symmetric matrices, then  $A^\top$ ,  $A+B$ ,  $A-B$ , and  $kA$  are all symmetric.
- For all  $m \times n$  matrices  $A$ ,  $A^\top A$  and  $AA^\top$  are both symmetric matrices.
- For all  $n \times n$  matrices  $A$ ,  $A + A^\top$  is a symmetric matrix, while  $A - A^\top$  is a skew-symmetric matrix.

**Theorem 6.** *If  $A$  and  $B$  are symmetric matrices, then  $AB$  is symmetric if and only if  $A$  and  $B$  commute.*

**Theorem 7.** *If  $A$  is an invertible symmetric matrix, then  $A^{-1}$  is also symmetric.*