MAT 260 LINEAR ALGEBRA
LECTURE 10

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4.2 — Subspaces

Definition 1. Let V be a vector space, with $\oplus$ and $\odot$ denoting its addition and scalar multiplication operations respectively. A **nonempty set** W is a **"subspace"** of V if all the following 3 properties are satisfied.

(1) $W \subseteq V$.
(2) The addition and scalar multiplication operations in W are **inherited from** V. In other words, if $\oplus_W : W \times W \to W$ and $\odot_W : \mathbb{R} \times W \to W$ denote the addition and scalar multiplication operations in W respectively, then
$$\text{for all } \mathbf{u}, \mathbf{v} \in W,\ \oplus_W(\mathbf{u}, \mathbf{v}) = \oplus(\mathbf{u}, \mathbf{v}), \text{ and}$$
$$\text{for all } k \in \mathbb{R}, \text{ and for all } \mathbf{v} \in W,\ \odot_W(k, \mathbf{v}) = \odot(k, \mathbf{v}).$$
(3) W, with $\oplus_W$ and $\odot_W$ defined on W as maps, is a vector space.

Question 2. What happens if we drop (2) from Definition 1?

Answer. Consider $V = \mathbb{R}$ with the standard vector addition and scalar multiplication. Consider $W = \mathbb{R}^+$, with addition and scalar multiplication on W defined such that
$$\mathbf{u} \oplus_W \mathbf{v} = u \cdot v \quad \text{and} \quad k \odot \mathbf{u} = u^k.$$
Recall from Example 8 and Problem 16 in Lecture Note 3 that W is a vector space. In other words, this choice of V and W satisfy conditions (1) and (3) in Definition 1. However, we don't like to call W to be a subspace of V. This is why condition (2) is necessary. □

If conditions (1) and (2) in Definition 1 are known to be satisified, it still takes a lot of work to verify condition (3), since verifying whether W is a vector space involves verifying all 10 axioms **for W**. Is there a quicker way?

Theorem 3. *Let V be a vector space, and let W be a nonempty subset V such that the addition and scalar multiplication in W are inherited from V. Then W is a subspace of V if and only if Axioms* (1) *and* (6) *hold* ***for W***.

Proof. If W is a subspace of V, then all 10 axioms hold for W. In particular, Axioms (1) and (6) hold. Hence, the "only if" direction is trivial.

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Now, assume that Axioms (1) and (6) hold **for W**.

Note: The proofs of Axioms (2), (3), (7), (8), (9), (10) **for W** are very similar since they are of the same type.

Axiom (2) **for W**:

For all $\mathbf{u}, \mathbf{v} \in \underline{W}$,

$$\begin{aligned} LHS &= \mathbf{u} \oplus_W \mathbf{v} \\ &= \mathbf{u} \oplus \mathbf{v} \quad \text{(since addition in } W \text{ is inherited from } V) \\ &= \mathbf{v} \oplus \mathbf{u} \quad \text{(by Axiom (2) } \textbf{for } V) \\ &= \mathbf{v} \oplus_W \mathbf{u} \quad \text{(since addition in } W \text{ is inherited from } V) \\ &= RHS. \end{aligned}$$

Therefore, Axiom (2) **for W** holds.

Axiom (3) **for W**:

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \underline{W}$,

$$\begin{aligned} LHS &= \mathbf{u} \oplus_W (\mathbf{v} \oplus_W \mathbf{w}) \\ &= \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) \quad \text{(since addition in } W \text{ is inherited from } V\text{, and } \mathbf{v} \oplus_W \mathbf{w} \in W) \\ &= (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \quad \text{(by Axiom (3) } \textbf{for } V) \\ &= (\mathbf{u} \oplus_W \mathbf{v}) \oplus_W \mathbf{w} \quad \text{(since addition in } W \text{ is inherited from } V\text{, and } \mathbf{u} \oplus_W \mathbf{v} \in W) \\ &= RHS. \end{aligned}$$

Therefore, Axiom (3) **for W** holds.

Axiom (7) **for W**:

For all $k \in \mathbb{R}$ and for all $\mathbf{u}, \mathbf{v} \in \underline{W}$,

$$\begin{aligned} LHS &= k \odot_W (\mathbf{u} \oplus_W \mathbf{v}) \\ &= k \odot (\mathbf{u} \oplus \mathbf{v}) \quad \text{(since addition and scalar multiplication in } W \text{ are inherited from } V\text{,} \\ &\qquad\qquad \text{and } \mathbf{u} \oplus_W \mathbf{v} \in W) \\ &= k \odot \mathbf{u} \oplus k \odot \mathbf{v} \quad \text{(by Axiom (7) } \textbf{for } V) \\ &= k \odot_W \mathbf{u} \oplus_W k \odot_W \mathbf{v} \quad \text{(since addition and scalar multiplication in } W \text{ are inherited} \\ &\qquad\qquad \text{from } V\text{, and } k \odot_W \mathbf{u} \text{ and } k \odot_W \mathbf{v} \text{ are in } W) \\ &= RHS. \end{aligned}$$

Therefore, Axiom (7) **for W** holds.

Axiom (8) **<u>for W</u>**:

For all $k, \ell \in \mathbb{R}$ and for all $\mathbf{v} \in \underline{W}$,

$$
\begin{aligned}
LHS &= (k+\ell) \odot_W \mathbf{v} \\
&= (k+\ell) \odot \mathbf{v} \qquad \text{(since scalar multiplication in } W \text{ is inherited from } V\text{)} \\
&= k \odot \mathbf{v} \oplus \ell \odot \mathbf{v} \qquad \text{(by Axiom (8) } \underline{\textbf{for } V}\text{)} \\
&= k \odot_W \mathbf{v} \oplus_W \ell \odot_W \mathbf{v} \qquad \text{(since addition and scalar multiplication in } W \text{ are inherited} \\
&\qquad\qquad \text{from } V\text{, and } k \odot_W \mathbf{v} \text{ and } \ell \odot_W \mathbf{v} \text{ are in } W\text{)} \\
&= RHS.
\end{aligned}
$$

Therefore, Axiom (8) **<u>for W</u>** holds.

Axiom (9) **<u>for W</u>**:

For all $k, \ell \in \mathbb{R}$ and for all $\mathbf{v} \in \underline{W}$,

$$
\begin{aligned}
LHS &= (k \cdot \ell) \odot_W \mathbf{v} \\
&= (k \cdot \ell) \odot \mathbf{v} \qquad \text{(since scalar multiplication in } W \text{ is inherited from } V\text{)} \\
&= k \odot (\ell \odot \mathbf{v}) \qquad \text{(by Axiom (9) } \underline{\textbf{for } V}\text{)} \\
&= k \odot_W (\ell \odot_W \mathbf{v}) \qquad \text{(since scalar multiplication in } W \text{ is inherited from } V\text{,} \\
&\qquad\qquad \text{and } \ell \odot_W \mathbf{v} \in W\text{)} \\
&= RHS.
\end{aligned}
$$

Therefore, Axiom (9) **<u>for W</u>** holds.

Axiom (10) **<u>for W</u>**:

For all $\mathbf{v} \in \underline{W}$,

$$
\begin{aligned}
LHS &= 1 \odot_W \mathbf{v} \\
&= 1 \odot \mathbf{v} \qquad \text{(since scalar multiplication in } W \text{ is inherited from } V\text{)} \\
&= \mathbf{v} \qquad \text{(by Axiom (10) } \underline{\textbf{for } V}\text{)} \\
&= RHS.
\end{aligned}
$$

Therefore, Axiom (10) **<u>for W</u>** holds.

Axiom (4) **<u>for W</u>**:

Since W is nonempty, pick $\mathbf{v} \in W$. Consider $0 \odot_W \mathbf{v}$. By Axiom (6) **<u>for W</u>**, $0 \odot_W \mathbf{v} \in W$.

$$
\begin{aligned}
0 \odot_W \mathbf{v} &= 0 \odot \mathbf{v} \qquad \text{(since scalar multiplication in } W \text{ is inherited from } V\text{)} \\
&= \mathbf{id} \qquad \text{(by Theorem A } \underline{\textbf{for } V} \text{ in Lecture Note 6).}
\end{aligned}
$$

Therefore, $\mathbf{id} \in W$, and Axiom (4) **<u>for W</u>** holds.

Axiom (5) **<u>for W</u>**:

For all $\mathbf{v} \in W$, consider $-1 \odot_W \mathbf{v}$. By Axiom (6) **<u>for W</u>**, $-1 \odot_W \mathbf{v} \in W$.

$$\begin{aligned} -1 \odot_W \mathbf{v} &= -1 \odot \mathbf{v} \qquad \text{(since scalar multiplication in } W \text{ is inherited from } V\text{)} \\ &= -\mathbf{v} \qquad \text{(by Theorem C } \underline{\textbf{for } V} \text{ in Lecture Note 6).} \end{aligned}$$

Therefore, for all $\mathbf{v} \in W$, $-\mathbf{v} \in W$, and Axiom (5) **<u>for W</u>** holds.

□

From now on, if we know that the addition and scalar multiplication in W are inherited from V, we will simply use $\oplus$ and $\odot$ to represent them. Furthermore, unless otherwise specified, the addition and scalar multiplication in any subset of V are assumed to be inherited from V.

Example 4. Let $V = \mathbb{R}^3$ with standard addition and scalar multiplication. The following subsets are subspaces of V.

- The set containing only the origin, i.e., $\{(0, 0, 0)\}$.
- A straight line passing through the origin, i.e., $\{t(a, b, c) : t \in \mathbb{R}\}$ for some $(a, b, c) \in \mathbb{R}^3$.
- A flat plane passing through the origin, i.e., $\{t_1(a_1, b_1, c_1) \oplus t_2(a_2, b_2, c_2) : t_1, t_2 \in \mathbb{R}\}$, where $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{R}^3$, and there do not exist real numbers k and ℓ satisfying $k(a_1, b_1, c_1) = \ell(a_2, b_2, c_2)$.
- $\mathbb{R}^3$.

Question 5. A straight line or a flat plane that does not pass through the origin is NOT a subspace of $\mathbb{R}^3$. Why?

Question 6. (5 bonus points if answered by next Friday) Are those subsets listed in Example 4 **<u>all</u>** the possible subspaces of $\mathbb{R}^3$? Why?

Example 7. Let $V = F(-\infty, \infty)$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$, as defined in Example 6 of Lecture Note 3. The following subsets are subspaces of V.

- The set of all "integrable" functions, i.e., $\{\mathbf{f} \in V : \int_{-\infty}^{\infty} \mathbf{f}(x)dx \text{ is well-defined}\}$.
- The set of all continuous functions, denoted by $C(-\infty, \infty)$.
- The set of all differentiable functions.
- The set of all functions whose first derivatives exist and are continuous, denoted by $C^1(-\infty, \infty)$.
- The set of all functions whose first n-th derivatives exist and are continuous, denoted by $C^n(-\infty, \infty)$.
- The set of all functions whose first n-th derivatives exist for all $n \in \mathbb{N}$, denoted by $C^\infty(-\infty, \infty)$.
- The set of all polynomials, denoted by P_∞.
- The set of all polynomials of degrees **<u>up to</u>** n, denoted by P_n.

Note: From the second bullet point onwards, the subspaces are arranged in the order that each subspace is contained in the previous one.

Question 8. Is the set of all polynomials of degree n a subspace of $F(-\infty, \infty)$?

Homework

Problem 9 (Textbook 4.2.1)**.** Let $V = \mathbb{R}^3$ be a vector space with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of V.

(a) The set of vectors of the form $(a, 0, 0)$, i.e., $W = \{(a, b, c) \in V : b = 0 \text{ and } c = 0\}$.
(b) The set of vectors of the form $(a, 1, 1)$.
(c) The set of vectors of the form (a, b, c), where $b = a + c$.
(d) The set of vectors of the form (a, b, c), where $b = a + c + 1$.
(e) The set of vectors of the form $(a, b, 0)$.

Problem 10 (Textbook 4.2.3, 4.2.4(d))**.** Let $V = P_3$ be the vector space of all polynomials with degrees **up to** 3, with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of V.

(a) The set of polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$, i.e., $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in V : a_0 = 0\}$.
(b) The set of polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
(c) The set of polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are integers.
(d) The set of polynomials $a_0 + a_1x$, where a_0 and a_1 are real numbers.
(e) The set of polynomials of degree 2.

Problem 11 (Textbook 4.2.4)**.** Let $V = F(-\infty, \infty)$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$, with standard addition and scalar multiplication. Use Theorem 3 of Lecture Note 8 to determine whether the following sets are subspaces of V.

(a) The set of functions f in $F(-\infty, \infty)$ for which $f(0) = 0$, i.e., $W = \{f \in V : f(0) = 0\}$.
(b) The set of functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.
(c) The set of functions f in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.

Problem 12. Let V be a vector space, and let W be a nonempty subset V such that the addition and scalar multiplication in W are inherited from V. Prove that W is a subspace of V if and only if for all $\mathbf{u}, \mathbf{v} \in W$, for all $k \in \mathbb{R}$, $\mathbf{u} \oplus k \odot \mathbf{v} \in W$.

Problem 13. Let V be a vector space. Let I be a nonempty set (often called the "index set"), and let W_i be a subspace of V for all $i \in I$. Prove that $\bigcap_{i \in I} W_i$, with addition and scalar multiplication inherited from V, is a subspace of V.

4.2 — Spanning

From this section onwards, we are no longer going to use $\oplus$ and $\odot$, and you need to able to tell from the context.

Question 14. Let V be a vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq V$. What kind of vectors in V can be created from S via addition and scalar multiplication?

Definition 15. Let V be a vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq V$. A **"linear combination"** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ (or a **"linear combination"** of S) is

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n,$$

where k_i's are scalars. (*Note*: This sum is well-defined because of Axiom (3) for V.)

The **"span"** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ (or a **"span"** of S) is the set of ALL linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, i.e.,

$$\text{span}(S) = \{\mathbf{v} \in V : \mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n, \text{ where } k_1, k_2, \dots, k_n \in \mathbb{R}\}.$$

If $S = \emptyset$, then we define $\text{span}(S) = \{\mathbf{id}\}$.

Theorem 16. *Let V be a vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq V$. Then* span$(S)$ *is the* ***"smallest"*** *subspace of V containing S.*

Proof. The few words "smallest subspace of V containing S" has actually hidden THREE important statements:

(1) span(S) is a subspace of V.
(2) span(S) contains S, i.e., $S \subseteq \text{span}(S)$.
(3) For all subspaces W of V containing S, we have W containing span(S). Equivalently, for all subspaces W of V such that $S \subseteq W$, we have $\text{span}(S) \subseteq W$.

Proof of (1): To show that span(S) is a subspace of V, we need to verify Axioms (1) and (6) **for span(S)**, according to Theorem 3.

Axiom (1) **for span(S)**:

For all $\mathbf{u}, \mathbf{v} \in$ **span(S)**,

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n \quad \text{and } \mathbf{v} = \ell_1\mathbf{v}_1 + \ell_2\mathbf{v}_2 + \cdots + \ell_n\mathbf{v}_n,$$

where $k_1, k_2, \dots, k_n, \ell_1, \ell_2, \dots, \ell_n \in \mathbb{R}$.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) + (\ell_1\mathbf{v}_1 + \ell_2\mathbf{v}_2 + \cdots + \ell_n\mathbf{v}_n) \\ &= (k_1 + \ell_1)\mathbf{v}_1 + (k_2 + \ell_2)\mathbf{v}_2 + \cdots + (k_n + \ell_n)\mathbf{v}_n \qquad \text{(by Axioms (2), (3), and (8) for } V) \\ &\in \text{span}(S),\end{aligned}$$

since $(k_1+\ell_1),(k_2+\ell_2),\ldots,(k_n+\ell_n)\in\mathbb{R}$. Therefore, Axiom (1) **for span**(S) holds.

Axiom (6) **for span**(S):

For all $a\in\mathbb{R}$ and for all $\mathbf{v}\in$ **span(S)**,

$$\mathbf{v}=\ell_1\mathbf{v}_1+\ell_2\mathbf{v}_2+\cdots+\ell_n\mathbf{v}_n,$$

where $\ell_1,\ell_2,\ldots,\ell_n\in\mathbb{R}$.

$$\begin{aligned} a\mathbf{v} &= a(\ell_1\mathbf{v}_1+\ell_2\mathbf{v}_2+\cdots+\ell_n\mathbf{v}_n)\\ &= a(\ell_1\mathbf{v}_1)+a(\ell_2\mathbf{v}_2)+\cdots+a(\ell_n\mathbf{v}_n) && \text{(by Axiom (7) for } V)\\ &= (a\ell_1)\mathbf{v}_1+(a\ell_2)\mathbf{v}_2+\cdots+(a\ell_n)\mathbf{v}_n && \text{(by Axiom (9) for } V)\\ &\in \operatorname{span}(S), \end{aligned}$$

since $a\ell_1,a\ell_2,\ldots,a\ell_n\in\mathbb{R}$. Therefore, Axiom (6) **for span**(S) holds.

Proof of (2): For all $i=1,2,\ldots,n$,

$$\begin{aligned} &0\mathbf{v}_1+0\mathbf{v}_2+\cdots+0\mathbf{v}_{i-1}+1\mathbf{v}_i+0\mathbf{v}_{i+1}+\cdots+0\mathbf{v}_n\\ =\ &\mathbf{id}+\cdots+\mathbf{id}+\mathbf{v}_i+\mathbf{id}+\cdots+\mathbf{id} \qquad \text{(by Theorem A and Axiom (10) for } V)\\ =\ &\mathbf{v}_i. \qquad \text{(by the definition of addition identity)} \end{aligned}$$

This shows that every $\mathbf{v}_i$ is an element of span(S). Therefore, $S\subseteq$ span(S).

Proof of (3): Let W be a subspace of V such that $S\subseteq W$. For all $\mathbf{v}\in$ span(S),

$$\mathbf{v}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n,$$

where $k_1,k_2,\ldots,k_n\in\mathbb{R}$.

Since $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\in W$, we have $k_1\mathbf{v}_1,k_2\mathbf{v}_2,\ldots,k_n\mathbf{v}_n\in W$ by Axiom (6) for W. Now, by Axiom (1) for W,

$$\begin{gathered} k_1\mathbf{v}_1+k_2\mathbf{v}_2\in W;\\ k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3=(k_1\mathbf{v}_1+k_2\mathbf{v}_2)+k_3\mathbf{v}_3\in W;\\ k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+k_4\mathbf{v}_4=(k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3)+k_4\mathbf{v}_4\in W;\\ \vdots\\ k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_{n-1}\mathbf{v}_{n-1}+k_n\mathbf{v}_n=(k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_{n-1}\mathbf{v}_{n-1})+k_n\mathbf{v}_n\in W. \end{gathered}$$

Hence, $\mathbf{v}\in W$. Therefore, span$(S)\subseteq W$.

□

Example 17. Let $V=\mathbb{R}^n$ with standard addition and scalar multiplication, and let $\mathbf{e}_1=(1,0,0,\ldots,0)$, $\mathbf{e}_2=(0,1,0,\ldots,0)$, $\ldots$, $\mathbf{e}_n=(0,0,0,\ldots,1)$. It is easy to see that span$\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}=\mathbb{R}^n$. These $\mathbf{e}_i$'s are called "**standard unit vectors**" in $\mathbb{R}^n$.

Example 18. Let $V=P_n$, the vector space of polynomials of degrees up to n, with standard addition and scalar multiplication. It is easy to see that span$\{1,x,x^2,\ldots,x^n\}=P_n$.

Theorem 19. *Let V be a vector space, and let S and T be two finite subsets of V. Then*

$$\operatorname{span}(S) = \operatorname{span}(T)$$

if and only if

$$S \subseteq \operatorname{span}(T) \text{ and } T \subseteq \operatorname{span}(S).$$

Proof. ($\Rightarrow$) Assume that $\operatorname{span}(S) = \operatorname{span}(T)$. Then by statement (2) in the proof of Theorem 16, $S \subseteq \operatorname{span}(S) = \operatorname{span}(T)$ and $T \subseteq \operatorname{span}(T) = \operatorname{span}(S)$.

($\Leftarrow$) Assume that $S \subseteq \operatorname{span}(T)$ and $T \subseteq \operatorname{span}(S)$. By statement (1) in the proof of Theorem 16, $\operatorname{span}(T)$ and $\operatorname{span}(S)$ are subspaces of V.

- Since $S \subseteq \operatorname{span}(T)$, by statement (3) in the proof of Theorem 16, $\operatorname{span}(S) \subseteq \operatorname{span}(T)$;
- since $T \subseteq \operatorname{span}(S)$, by statement (3) in the proof of Theorem 16, $\operatorname{span}(T) \subseteq \operatorname{span}(S)$.

Therefore, $\operatorname{span}(S) = \operatorname{span}(T)$.

□

Example 20. Let $V = \mathbb{R}^3$ with standard addition and scalar multiplication, and let $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$. Determine whether $\{\mathbf{u}, \mathbf{v}\}$ spans $\mathbf{x} = (9, 2, 7)$ and $\mathbf{y} = (4, -1, 8)$.

Solution. Let $k_1, k_2 \in \mathbb{R}$ such that $k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{x}$. In other words,

$$k_1(1, 2, -1) + k_2(6, 4, 2) = (9, 2, 7).$$

From this equation, we get the following **system of linear equations**.

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

We can solve this system by the **method of elimination**, which involves

- interchanging any two equations,
- multiplying an equation by a nonzero constant, and
- adding a constant multiple of an equation to another equation.

The goal is to get 0s for the coefficients for some variables.

Here, we are going to convert the system of linear equations into an **augmented matrix**, and we are going to perform **Gauss-Jordan elimination**, which is composed of a sequence of **elementary row operations**:

- Interchange the i-th and j-th row ($R_i \leftrightarrow R_j$).
- Multiply the i-th row by a nonzero constant (cR_i).
- Add a constant multiple of the j-th row to the i-th row ($R_i + cR_j$).

The goal is to get to the **reduced row echelon form**:

- Every row of all zeros are at the bottom of the matrix.
- For every row which are not all zeros, the first nonzero entry is 1, called the **leading 1**.
- The positions of leading 1s are going downward and rightward.
- All entries in the same column as a leading 1 are all zeros.

$$\left(\begin{array}{rr|r} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{rr|r} 1 & 6 & 9 \\ 0 & -8 & -16 \\ -1 & 2 & 7 \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{rr|r} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{8}R_2} \left(\begin{array}{rr|r} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{array}\right) \xrightarrow{R_3-8R_2} \left(\begin{array}{rr|r} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

This finishes the **forward phase** of the Gauss-Jordan elimination, and will result in a **row echelon form** (which is the same as reduced row echelon form, except the last condition is removed.) Next, we have the **backward phase** of the Gauss-Jordan elimination.

$$\left(\begin{array}{rr|r} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-6R_2} \left(\begin{array}{rr|r} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

From the reduced row echelon form, the first row represents the equation $k_1 = -3$, and the second row represents $k_2 = 2$, so the solutions to the system of linear equations are

$$(k_1, k_2) = (-3, 2).$$

Therefore, $\mathbf{x}$ is spanned by $\{\mathbf{u}, \mathbf{v}\}$.

Similarly, let $\ell_1, \ell_2 \in \mathbb{R}$ such that $\ell_1\mathbf{u} + \ell_2\mathbf{v} = \mathbf{y}$. In other words,

$$\ell_1(1, 2, -1) + \ell_2(6, 4, 2) = (4, -1, 8).$$

From this equation, we get the following system of linear equations.

$$\begin{aligned} \ell_1 + 6\ell_2 &= 4 \\ 2\ell_1 + 4\ell_2 &= -1 \\ -\ell_1 + 2\ell_2 &= 8 \end{aligned}$$

$$\left(\begin{array}{rr|r} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & -8 & -9 \\ -1 & 2 & 8 \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{8}R_2} \left(\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 8 & 12 \end{array}\right) \xrightarrow{R_3-8R_2} \left(\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 0 & 3 \end{array}\right) \xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 0 & 1 \end{array}\right)$$

From this row echelon form, the last row represents the equation $0 = 1$, which is impossible, so there are no solutions to the system of linear equations. Therefore, $\mathbf{y}$ is not spanned by $\{\mathbf{u}, \mathbf{v}\}$.

□

Example 21. Let $V = P_2$ with standard addition and scalar multiplication, and let $\mathbf{v}_1 = 1 + x + 2x^2$, $\mathbf{v}_2 = 1 + x^2$ and $\mathbf{v}_3 = 3 + 4x + 7x^2$.

(a) Express $\mathbf{w} = 4 + 3x + 7x^2$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(b) Determine whether $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = P_2$.

Solution. Let $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{w}$. In other words,

$$k_1(1 + x + 2x^2) + k_2(1 + x^2) + k_3(3 + 4x + 7x^2) = 4 + 3x + 7x^2.$$

From this equation, we get the following system of linear equations.

$$\begin{aligned} k_1 + k_2 + 3k_3 &= 4 \\ k_1 \qquad + 4k_3 &= 3 \\ 2k_1 + k_2 + 7k_3 &= 7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 1 & 0 & 4 & 3 \\ 2 & 1 & 7 & 7 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R_1-R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since there is no leading 1 in the column corresponding to the variable k_3, we assign a **free parameter** $k_3 = t$, where $t \in \mathbb{R}$.

- The first row represents the equation $k_1 + 4t = 3$, so $k_1 = -4t + 3$; and
- the second row represents the equation $k_2 - t = 1$, so $k_2 = t + 1$.

So, the solutions to the system of linear equations are

$$(k_1, k_2, k_3) = (-4t + 3, t + 1, t),$$

where $t \in \mathbb{R}$ is a free parameter. Therefore, $(-4t + 3)\mathbf{v}_1 + (t + 1)\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{w}$ for all $t \in \mathbb{R}$.

Let $\mathbf{x} = b_0 + b_1x + b_2x^2$, and let $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$ such that $\ell_1\mathbf{v}_1 + \ell_2\mathbf{v}_2 + \ell_3\mathbf{v}_3 = \mathbf{x}$. In other words,

$$\ell_1(1 + x + 2x^2) + \ell_2(1 + x^2) + \ell_3(3 + 4x + 7x^2) = b_0 + b_1x + b_2x^2.$$

From this equation, we get the following system of linear equations.

$$\begin{aligned} \ell_1 + \ell_2 + 3\ell_3 &= b_0 \\ \ell_1 \qquad + 4\ell_3 &= b_1 \\ 2\ell_1 + \ell_2 + 7\ell_3 &= b_2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & b_0 \\ 1 & 0 & 4 & b_1 \\ 2 & 1 & 7 & b_2 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & b_0 \\ 0 & -1 & 1 & b_1 - b_0 \\ 0 & -1 & 1 & b_2 - 2b_0 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & b_0 \\ 0 & 1 & -1 & b_0 - b_1 \\ 0 & -1 & 1 & b_2 - 2b_0 \end{array}\right)$$

$$\xrightarrow[R_1-R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & b_1 \\ 0 & 1 & -1 & b_0 - b_1 \\ 0 & 0 & 0 & b_2 - b_0 - b_1 \end{array}\right)$$

The third row represents the equation $0 = b_2 - b_0 - b_1$. If $b_2 - b_0 - b_1 \neq 0$, then there is no solution to the system of linear equations. Therefore, there exists $\mathbf{x} \in P_2$ that cannot be spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, i.e., $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \neq P_2$.

□