MA 1C (SECTION 11) RECITATION 9

TONY WING HONG WONG

1. Surface Integral

This is an analogue of line integral.
Let U be simply connected and ∂U be piecewise C^1 Jordan curve. Let $\Phi : \overline{U} \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be C^1. Φ is *simple* if Φ is one-one (injective), and Φ is *regular* at (u_0, v_0) if $\frac{\partial \Phi}{\partial u}(u_0, v_0) \times \frac{\partial \Phi}{\partial v}(u_0, v_0) \neq 0$. In fact, $\frac{\partial \Phi}{\partial u}(u_0, v_0) \times \frac{\partial \Phi}{\partial v}(u_0, v_0)$ is a normal vector at (u_0, v_0) to the surface $\Phi(U)$.

If Φ is simple and regular, then $\text{Area}(\Phi(U)) = \iint_D \|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\| \, dudv$.
Surface integral for scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ is
$\iint_\Phi f dS = \iint_D f(\Phi(u, v))\|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\| \, dudv$.
Surface integral for vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ is
$\iint_\Phi F \cdot \mathbf{n} dS = \iint_D F(\Phi(u, v)) \cdot \mathbf{n}(u, v)\|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\| \, dudv = \iint_D F(\Phi(u, v)) \cdot (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}) \, dudv$,
where $\mathbf{n}(u, v)$ is the unit normal vector at (u, v) defined by $(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})/\|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|$.

Let $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ and $\Phi(u, v) = (X(u, v), Y(u, v), Z(u, v))$. Let $\frac{\partial(Y,Z)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} \end{pmatrix}$. Then $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = \left(\frac{\partial(Y,Z)}{\partial(u,v)}, \frac{\partial(Z,X)}{\partial(u,v)}, \frac{\partial(X,Y)}{\partial(u,v)}\right)$, and
$\iint_S P(x, y, z) \, dy \wedge dz = \iint_D P(\Phi(u, v)) \frac{\partial(Y,Z)}{\partial(u,v)} \, dudv$. Furthermore,
$\iint_\Phi F \cdot \mathbf{n} dS = \iint_S P(x, y, z) \, dy \wedge dz + Q(x, y, z) \, dz \wedge dx + R(x, y, z) \, dx \wedge dy$.

2. Stokes Theorem

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a positively oriented piecewise C^1 Jordan curve, $C = \gamma([a, b])$. Let U be the interior of C, and let $D = U \cup C$. Let $\Phi : D \to \mathbb{R}^3$ be C^2, simple and regular parametrization. Let F be a C^1 vector field defined on $\Phi(D)$. Let $\alpha = \Phi \circ \gamma$. Then
$\iint_\Phi (\nabla \times F) \cdot \mathbf{n} \, dS = \int_{\alpha([a,b])} F \cdot d\alpha$.

3. Divergence Theorem

Let V be a solid bounded by an orientable closed surface S, and $F : V \to \mathbb{R}^3$ is a C^1 vector field, then
$\iiint_V \nabla \cdot F \, dxdydz = \iint_S F \cdot \mathbf{n} \, dS$.

Date: May 30, 2013.

4. Examples

Example 1. The cylinder $x^2 + y^2 = 2x$ cuts out a portion of a surface S from the upper nappe of the cone $x^2 + y^2 = z^2$. Compute the value of the surface integral

$$\iint_S (x^4 - y^4 + y^2z^2 - z^2x^2 + 1)\, dS.$$

Solution. Let $\Phi(x, y) = (x, y, \sqrt{x^2 + y^2})$ (since we only need the positive branch for z), and let $f(x, y, z) = x^4 - y^4 + y^2z^2 - z^2x^2 + 1$. Then $f(\Phi(x, y)) = 1$. Note that $\frac{\partial \Phi}{\partial x} = \big(1, 0, \frac{x}{\sqrt{x^2+y^2}}\big)$, $\frac{\partial \Phi}{\partial y} = \big(0, 1, \frac{y}{\sqrt{x^2+y^2}}\big)$, so $\|\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y}\| = \Big\|\big(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\big)\Big\| = \sqrt{2}$. Therefore, the surface integral becomes $\iint_S 1 \cdot \sqrt{2}\, dxdy = \sqrt{2}\pi$. □

Example 2. Transform the following surface integral $\iint_S (\text{curl} F) \cdot \mathbf{n}\, dS$ to a line integral by Stokes theorem, and then evaluate the line integral.
$F(x, y, z) = (y, z, x)$, and S is the portion of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$, and $\mathbf{n}$ is the unit normal vector with a nonnegative z-component.

Solution. Let C be the unit circle on the xy-plane. Then C is the boundary of S, and it can be parametrized by $\alpha(t) = (\cos t, \sin t, 0)$. By Stokes theorem, the surface integral becomes $\int_C F \cdot\, d\alpha = \int_0^{2\pi} (\sin t, 0, \cos t) \cdot (-\sin t, \cos t, 0)\, dt = \int_0^{2\pi} \frac{\cos 2t - 1}{2}\, dt = \frac{\sin 2t}{4} - \frac{t}{2}\Big|_0^{2\pi} = -\pi.$ □