MA 1C (SECTION 11) RECITATION 8

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1. Change of Variables

Let $\phi : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ be a diffeomorphism, i.e. ϕ is bijective and both ϕ and ϕ^{-1} are in C^1. Let $f : V \to \mathbb{R}$ be integrable. Then $f \circ \phi : U \to \mathbb{R}$ is integrable and
$\int_V f(x_1, \dots, x_n)\, dx_1 \dots dx_n = \int_U f(\phi(\mathbf{u}))\, |\det D\phi(\mathbf{u})|\; du_1 \dots du_n$,
where $\det D\phi(\mathbf{u}) = \begin{vmatrix} \frac{\partial \phi_1}{\partial u_1} & \cdots & \frac{\partial \phi_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial u_1} & \cdots & \frac{\partial \phi_n}{\partial u_n} \end{vmatrix}$.

1. If $\phi(\mathbf{u}) = A\mathbf{u}$ for some invertible matrix A, then
$\int_V f(x_1, \dots, x_n)\, dx_1 \dots dx_n = \int_U f(\phi(\mathbf{u}))\, |\det A|\; du_1 \dots du_n$.
2. If $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(r, \theta) \mapsto (x, y)$ by $x = r\cos\theta$ and $y = r\sin\theta$, then
$\int_V f(x, y)\, dxdy = \int_U f(r\cos\theta, r\sin\theta)\, rdrd\theta$.
3. If $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $(r, \theta, z) \mapsto (x, y, z)$ by $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$, then
$\int_V f(x, y, z)\, dxdydz = \int_U f(r\cos\theta, r\sin\theta, z)\, rdrd\theta d\phi$.
4. If $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $(r, \theta, \phi) \mapsto (x, y, z)$ (note that those two ϕ's are different) by $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$, then
$\int_V f(x, y, z)\, dxdydz = \int_U f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)\, r^2\sin\phi drd\theta d\phi$.

2. Surface Integral

This is an analogue of line integral.
Let U be simply connected and ∂U be piecewise C^1 Jordan curve. Let $\Phi : \overline{U} \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be C^1. Φ is *simple* if Φ is one-one (injective), and Φ is *regular* at (u_0, v_0) if $\frac{\partial \Phi}{\partial u}(u_0, v_0) \times \frac{\partial \Phi}{\partial v}(u_0, v_0) \neq 0$. In fact, $\frac{\partial \Phi}{\partial u}(u_0, v_0) \times \frac{\partial \Phi}{\partial v}(u_0, v_0)$ is a normal vector at (u_0, v_0) to the surface $\Phi(U)$.

If Φ is simple and regular, then $\text{Area}(\Phi(U)) = \iint_D \|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|\, dudv$.
Surface integral for scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ is
$\iint_\Phi f dS = \iint_D f(\Phi(u, v)) \|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|\, dudv$.
Surface integral for vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ is
$\iint_\Phi F \cdot \mathbf{n} dS = \iint_D F(\Phi(u, v)) \cdot \mathbf{n}(u, v) \|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|\, dudv = \iint_D F(\Phi(u, v)) \cdot (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})\, dudv$,
where $\mathbf{n}(u, v)$ is the unit normal vector at (u, v) defined by $(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}) / \|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|$.
Note that surface integrals have the same values under the change of parametrization.

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3. Examples

Example 1. Make a sketch of $S = \{(x, y) : 0 \leq y < 1 - x, 0 \leq x \leq 1\}$ and express the double integral $\iint_S f(x, y)dxdy$ as an iterated integral in polar coordinates.

Solution. S is the right-angled triangle with vertices $(1, 0)$, $(0, 1)$ and $(0, 0)$, so $0 \leq \theta \leq \frac{\pi}{2}$. For each θ, the ray $\{(x, (\tan\theta)x) : x \in \mathbb{R}^+\}$ intersects $y = 1 - x$ when $(\tan\theta)x = 1 - x$, or $x = 1/(1 + \tan\theta)$, so $0 \leq r \leq r_\theta$, where $r_\theta = \sqrt{x^2 + (1 - x)^2} = \sqrt{\frac{1}{(1+\tan\theta)^2} + \frac{\tan^2\theta}{(1+\tan\theta)^2}} = \frac{\sec\theta}{1+\tan\theta} = \frac{1}{\sin\theta+\cos\theta}$ since $\sec\theta$ and $1 + \tan\theta$ are positive in the range. Hence, the double integral $\iint_S f(x, y)dxdy = \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta+\cos\theta}} f(r\cos\theta, r\sin\theta)\, rdrd\theta$. □

Example 2. Evaluate the integral $\iiint_S (y^2 + z^2)dxdydz$ using cylindrical coordinates, where S is a right circular cone of altitude h, with its base of radius a, in the xy-plane and its axis along the z-axis.

Solution. By using cylindrical coordinates, we have $\int_0^a \int_0^{2\pi} \int_0^{h-\frac{hr}{a}} ((r\sin\theta)^2 + z^2)\, rdzd\theta dr = \int_0^a \int_0^{2\pi} r^3 \sin^2\theta(h - \frac{hr}{a}) + (h - \frac{hr}{a})^3 r/3 \, d\theta dr = \int_0^a \int_0^{2\pi} r^3(h - \frac{hr}{a})\frac{1-\cos 2\theta}{2} + (h - \frac{hr}{a})^3 r/3 \, d\theta dr = \pi \int_0^a hr^3 - \frac{h}{a}r^4 + \frac{2h^3}{3}r - \frac{2h^3}{a}r^2 + \frac{2h^3}{a^2}r^3 - \frac{2h^3}{3a^3}r^4 \, dr = \pi(\frac{ha^4}{4} - \frac{ha^5}{5a} + \frac{h^3a^2}{3} - \frac{2h^3a^3}{3a} + \frac{h^3a^4}{2a^2} - \frac{2h^3a^5}{15a^3}) = \frac{ha^2\pi}{60}(15a^2 - 12a^2 + 20h^2 - 40h^2 + 30h^2 - 8h^2) = \frac{ha^2\pi}{60}(3a^2 + 2h^2)$. □