MA 1C (SECTION 11) RECITATION 7

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1. Conservative Vector Fields

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, D open, f continuous. f is conservative if there is a C^1 $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $f = \nabla\phi$.
f is conservative if and only if $\int_C f \cdot d\alpha = 0$ for all piecewise C^1 closed curve C in D.

If D is connected and f is conservative, then ϕ is uniquely determined up to a constant, namely $\phi(x) = \int_C f \cdot d\alpha$, where C is any path from a fixed point $x_0 \in D$ to $x \in D$.

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, D open, f is C^1. f is conservative implies $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all i, j.
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, D open and simply connected (no holes), f is C^1. $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all i, j implies f is conservative.

2. Green's Theorem

Jordan curve in $\mathbb{R}^n$ is a simple closed curve (no self-intersection). In $\mathbb{R}^2$, we can talk about positively and negatively oriented.
In $\mathbb{R}^2$, the normal vector $\mathbf{n}(t) = \left(\frac{\alpha'(t)}{\|\alpha(t)\|}\right)'$ points outwards.

Green's Theorem:
$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, D open and simply connected, f is C^1, $\alpha : [a, b] \to D$ positively oriented piecewise C^1 Jordan curve. Then
$\int_C f \cdot d\alpha = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy$,
where R is the closed region bounded by the Jordan curve α.
It can also be written as $\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$, where $f = (P, Q)$.

Note that this integral does not depend on the parametrization α, but only on the curve C itself.

If R has holes, each has boundary a Jordan curve, then Green's Theorem implies
$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_C Pdx + Qdy - \int_{C_1} Pdx + Qdy - \cdots - \int_{C_k} Pdx + Qdy$.

Application for calculating areas:
$\text{Area}(R) = \iint_R 1dxdy = \frac{1}{2}\int_C xdy - ydx = \frac{1}{2}\int_a^b x(t)y'(t) - y(t)x'(t)dt$.

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Winding number:
Let $z_0 = (x_0, y_0)$ not on the curve $C = \alpha([a, b])$.
$W(\alpha, z_0) = \frac{1}{2\pi} \int_C -\frac{y-y_0}{r^2} dx + \frac{x-x_0}{r^2} dy = \frac{1}{2\pi} \int_a^b \frac{(x(t)-x_0)y'(t)-(y(t)-y_0)x'(t)}{(x(t)-x_0)^2+(y(t)-y_0)^2} dt.$
If α is a Jordan curve, then $W(\alpha, z_0) = \pm 1$ if z_0 is in the interior and $W(\alpha, z_0) = 0$ if z_0 is in the exterior.

3. Div, curl and grad

Grad:
$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

Div:
$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $\nabla \cdot f = \frac{\partial f}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}$ = trace of Jacobian matrix of f.

Laplacian:
$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $\Delta f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x_1^2} + \ldots, \frac{\partial^2 f}{\partial x_n^2}$ = trace of Hessian matrix of f.
f is harmonic if $\Delta f = 0$, a zero function.

Curl:
$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, $f = (P, Q, R)$, $\operatorname{curl} f = \nabla \times f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$.

4. Examples

Example 1. If $f(x, y) = (P(x), Q(y))$, then $\frac{\partial Q}{\partial x} = 0 = \frac{\partial P}{\partial y}$, so f is conservative and $\int_C f \cdot d\alpha = 0$ for all Jordan curves α.

Example 2. If $P(x, y) = xe^{-y^2}$ and $Q(x, y) = -x^2 ye^{-y^2} + \frac{1}{x^2+y^2}$, evaluate the line integral $\int P dx + Q dy$ around the boundary of the square of side $2a$ determined by the inequalities $|x| \leq a$ and $|y| \leq a$.

Solution. Note that if $Q'(x, y) = -x^2 ye^{-y^2}$, then $\frac{\partial Q'}{\partial x} = -2xye^{-y^2} = \frac{\partial P}{\partial y}$, so $f = (P, Q')$ is conservative, which gives $\int P dx + Q dy = \big(\int P dx + Q' dy \big) + \int \frac{1}{x^2+y^2} dy = \int \frac{1}{x^2+y^2} dy$, which is equal to $\int_a^{-a} \frac{1}{(-a)^2+y^2} dy + \int_{-a}^a \frac{1}{a^2+y^2} dy = 0$ since they cancel each other. $\square$