MA 1C (SECTION 11) RECITATION 6

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1. Mid-term statistics

		Whole class	Our section
HW	Average	79.31	81.76
	75-percentile	87.24	87.24
	Median	80.61	82.91
	25-percentile	73.98	77.42
Mid-term	Average	58.11	60.92
	75-percentile	68	69.5
	Median	57	62
	25-percentile	45.5	51
Overall	Average	65.65	69.25
	75-percentile	73.56	74.96
	Median	65.65	67.69
	25-percentile	58.85	64.27

2. Line integral

Let $\alpha : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 curve. Arc length parameter is
$s(t) = \int_a^t \|\alpha'(u)\| du$, so by fundamental theorem of calculus, $ds = \|\alpha'(t)\| dt$.

Let the path $\alpha([a, b]) = C$. If $C \subseteq D$, and $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then the line integral of f is
$\int_C f \cdot d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t) dt$.

If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, then the line integral with respect to arc length is
$\int_C f ds = \int_a^b f(\alpha(t)) \|\alpha'(t)\| dt$.

Connection between these two integrals:
Let $T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$, the unit tangent vector of the curve α. If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, define $g(\alpha(t)) = f(\alpha(t)) \cdot T(t)$. Then
$\int_C g ds = \int_a^b f(\alpha(t)) \cdot T(t) \|\alpha'(t)\| dt = \int_a^b f(\alpha(t)) \cdot \alpha(t) dt = \int_C f \cdot d\alpha$.

Change of parametrization of the line integral with respect to arc length:
Let $u : [c, d] \to [a, b]$ be a bijection, C^1 and $u'(t) \neq 0$ for all $t \in [c, d]$. Define the curve β such that $\beta(t) = \alpha(u(t))$.
If u is orientation preserving, then $u(c) = a$, $u(d) = b$ and $u' > 0$.

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Then $\int_c^d f(\beta(t))\|\beta'(t)\|dt = \int_c^d f(\alpha(u(t)))\|\alpha'(u(t))u'(t)\|dt = \int_c^d f(\alpha(u(t)))\|\alpha'(u(t))\|u'(t)dt$
$= \int_a^b f(\alpha(u))\|\alpha'(u)\|du$.
If u is orientation reversing, then $u(c) = b$, $u(d) = a$, and $u' < 0$.
Then $\int_c^d f(\beta(t))\|\beta'(t)\|dt = \int_c^d f(\alpha(u(t)))\|\alpha'(u(t))u'(t)\|dt = \int_c^d f(\alpha(u(t)))\|\alpha'(u(t))\|(-u'(t))dt$
$= -\int_b^a f(\alpha(u))\|\alpha'(u)\|du = \int_a^b f(\alpha(u))\|\alpha'(u)\|du$.

First fundamental theorem of calculus for line integrals:
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous vector field such that for every piecewise C^1 curve, the line integral of f depends only on the endpoints. Then $f = \nabla\phi$ for some C^1 scalar field $\phi : D \subseteq \mathbb{R}^n \to \mathbb{R}$.

Second fundamental theorem of calculus for line integrals:
Let $\alpha : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 curve. Let $\alpha([a, b]) = C \subseteq D$, $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, then $\int_C \nabla f \cdot d\alpha = f(\alpha(b)) - f(\alpha(a))$.

3. Connectedness

$S \subseteq \mathbb{R}^n$ is *connected*
if there does not exist disjoint open sets (in $\mathbb{R}^n$) U and V such that $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$ and $S \subseteq U \cup V$, or
if there does not exist a 'clopen' set in S except S and $\emptyset$. (Here, 'clopen' means closed and open at the same time with respect to the 'relative topology' of S.)

For a general set $S \subseteq \mathbb{R}^n$, S path-connected $\Rightarrow$ S connected.
For an open set $U \subseteq \mathbb{R}^n$, U connected $\Rightarrow$ U path-connected.
However, for a general set $S \subseteq \mathbb{R}^n$, S connected $\not\Rightarrow$ S path-connected.
e.g. $S = (\{(x, \sin(1/x)) : x \in \mathbb{R}\} \cup$ y-axis).

4. Examples

Example 1. Find $\int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy$, where C is a path from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$.

Solution. Let $\alpha(t) = (x(t), y(t)) = (t, t^2)$, and let $f(x, y) = (x^2 - 2xy, y^2 - 2xy)$. Then our integral is $\int_C f \cdot d\alpha = \int_{-1}^{2} \left(t^2 - 2(t)(t^2), (t^2)^2 - 2(t)(t^2)\right) \cdot (1, 2t)dt = \int_{-2}^{1} t^2 - 2t^3 + 2t^5 - 4t^4 dt$
$= \frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5}\Big|_{-2}^{1} = -36.9$.

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