MA 1C (SECTION 11) RECITATION 4

TONY WING HONG WONG

1. Hessian Test

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^2 function, i.e. the second derivative exists and is continuous. The **Hessian matrix** at $\mathbf{a} \in D$ is

$$H(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{a})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{a})}{\partial x_n \partial x_n} \end{pmatrix}.$$

Now, assume that $\nabla f(\mathbf{a}) = \mathbf{0}$. Then
if $H(\mathbf{a})$ is positive definite, $\mathbf{a}$ is a relative minimum;
if $H(\mathbf{a})$ is negative definite, $\mathbf{a}$ is a relative maximum;
if $H(\mathbf{a})$ is has both positive and negative eigenvalues, then $\mathbf{a}$ is a saddle point.

A symmetric matrix A is positive definite if
- all eigenvalues of A are strictly positive, or
- $\mathbf{v}^\top A\mathbf{v} > 0$ for all nonzero column vector $\mathbf{v}$, or
- all the leading principal submatrices have positive determinants.

A symmetric matrix A is negative definite if
- all eigenvalues of A are strictly negative, or
- $\mathbf{v}^\top A\mathbf{v} < 0$ for all nonzero column vector $\mathbf{v}$, or
- the leading principal submatrices have determinants of alternating signs, starting with negative.

2. Lagrange Multiplier

Let $f, g_1, \ldots, g_m : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be in C^1, and let $S = \{\mathbf{x} \in D : g_1(\mathbf{x}) = 0, \ldots, g_m(\mathbf{x}) = 0\}$, where $m < n$. If $\mathbf{a}$ is a local extremum of f in S and $\nabla g_1(\mathbf{a}), \ldots, \nabla g_m(\mathbf{a})$ are linear independent, then $\nabla f(\mathbf{a}) = \lambda_1 \nabla g_1(\mathbf{a}) + \cdots + \lambda_m \nabla g_m(\mathbf{a})$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

We can try to understand the method of Lagrange multiplier pictorially.

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3. Examples

Example 1. Minimize $f(x,y,z) = x^2+y^2+z^2$ subject to $g_1(x,y,z) = x^2+y^2-z^2 = 0$ and $g_2(x,y,z) = x+y+z-3 = 0$.

Solution. (1) Argue using compact sets (closed and bounded sets) to show that there is a global minimum for f on S.
(2) Find $\nabla g_1(x,y,z)$ and $\nabla g_2(x,y,z)$, and determine all (x,y,z) such that these two gradients are linearly independent (in this example, they are always linearly independent on $S = \{(x,y,z) : g_1(x,y,z) = g_2(x,y,z) = 0\}$).
(3) Use the method of Lagrange multiplier to obtain five equations and solve them. These five equations are

$$x^2+y^2-z^2 = 0,\ x+y+z-3 = 0,$$
$$2x = 2\lambda_1 x+\lambda_2,\ \ 2y = 2\lambda_1 y+\lambda_2,\ \ 2z = -2\lambda_1 z+\lambda_2.$$

□

Example 2. Assume $\frac{\partial g}{\partial z}(a,b,c) \neq 0$. Let $\alpha(t) = (a+tu_1, b+tu_2, h(a+tu_1, b+tu_2))$ be a path on the level surface $g(x,y,z) = 0$, where h comes from the implicit function theorem such that near (a,b,c), $g(x,y,z) = 0$ is the same as $z = h(x,y)$. Let (u_1,u_2,u_3) be on the tangent plane to the level surface $g(x,y,z) = 0$ at (a,b,c), i.e. $\nabla g(a,b,c)\cdot(u_1,u_2,u_3) = 0$. Show that $\alpha'(0) = (u_1,u_2,u_3)$.

Solution. By chain rule, $\alpha'(0) = (u_1, u_2, \frac{\partial h}{\partial x}(a,b)u_1 + \frac{\partial h}{\partial y}(a,b)u_2)$, so our goal is to show that $u_3 = \frac{\partial h}{\partial x}(a,b)u_1 + \frac{\partial h}{\partial y}(a,b)u_2$.
Note that $g(x,y,h(x,y)) = 0$ near (a,b,c), so by differentiating with respect to x and y at (a,b), we get

$$\frac{\partial g}{\partial x}(a,b,c) + \frac{\partial g}{\partial z}(a,b,c)\frac{\partial h}{\partial x}(a,b) = 0,$$
$$\frac{\partial g}{\partial y}(a,b,c) + \frac{\partial g}{\partial z}(a,b,c)\frac{\partial h}{\partial y}(a,b) = 0.$$

Recall that $\nabla g(a,b,c)\cdot(u_1,u_2,u_3) = 0$, so $\frac{\partial g}{\partial x}(a,b,c)u_1 + \frac{\partial g}{\partial y}(a,b,c)u_2 + \frac{\partial g}{\partial z}(a,b,c)u_3 = 0$. Therefore, we have $-\frac{\partial g}{\partial z}(a,b,c)\frac{\partial h}{\partial x}(a,b)u_1 - \frac{\partial g}{\partial z}(a,b,c)\frac{\partial h}{\partial y}(a,b)u_2 + \frac{\partial g}{\partial z}(a,b,c)u_3 = 0$. By dividing $\frac{\partial g}{\partial z}(a,b,c)$, we achieve our goal. □