MA 1C (SECTION 11) RECITATION 2

TONY WING HONG WONG

1. Total Differentiation

$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in \text{int}(D)$ if and only if there exists linear $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})\|_m}{\|\mathbf{h}\|_n} = 0.$$

The total derivative of f at $\mathbf{a}$, denoted as $T_{\mathbf{a}}f$ or $f'(\mathbf{a})$, is L. Each different $\mathbf{a}$ may give a different L.

Directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{u}$ is $T_a f(\mathbf{u})$ or $f'(\mathbf{a})(\mathbf{u})$, the matrix multiplication of $T_{\mathbf{a}}f$ to $\mathbf{u}$. It can also be denoted as $f'(\mathbf{a};\mathbf{u})$. Note that $\mathbf{a} \in D \subseteq \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^n$, and $f'(\mathbf{a};\mathbf{u}) \in \mathbb{R}^m$.

Equivalently, we can define the directional derivative as

$$f'(\mathbf{a};\mathbf{u}) = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t}.$$

$f'(\mathbf{a};\mathbf{u})$ exists if and only if $f_i'(\mathbf{a};\mathbf{u})$ exist, and $f'(\mathbf{a};\mathbf{u}) = (f_1'(\mathbf{a};\mathbf{u}), \ldots, f_m'(\mathbf{a};\mathbf{u}))$.

Partial derivatives are special cases of directional derivatives, using standard unit vectors $\mathbf{e}_i$. We find that $f'(\mathbf{a};\mathbf{e}_i) = \frac{\partial f}{\partial x_i} = (\frac{\partial f_m}{\partial x_i}, \ldots, \frac{\partial f_m}{\partial x_i})$. (Theoretically, we should write it as a column vector, since it is the column vector $T_{\mathbf{a}}f(\mathbf{e}_i)$.)

$$T_a f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$ **provided that total derivative exists**.

Total derivative $T_f(a)$ exists implies all directional derivative exists, but the converse is not true.
All partial derivatives exist in D and continuous at a implies total derivative $T_f(a)$ exists, but the converse is not true.

Gradient $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is only for $f : \mathbb{R}^n \to \mathbb{R}$. In fact, gradient is just the total derivative, $\nabla f(\mathbf{a}) = T_{\mathbf{a}}f$.

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If $f : \mathbb{R}^n \to \mathbb{R}$, $f'(\mathbf{a};\mathbf{u})$ is a real number instead of a vector. Its maximum value is $\|\nabla f(a)\|$, and the corresponding $\mathbf{u}$ is in the direction of $\nabla f(a)$.

$f : \mathbb{R} \to \mathbb{R}^m$ is a walk in the space, and we can call $f'(t) = (f_1'(t), \ldots, f_m'(t))$ the velocity vector.

2. Examples

Example 1. Let $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and determine if f is differentiable (i.e. has total derivative) at $(0,0)$.

Solution. When $x \neq 0$, $\frac{\partial f}{\partial x} = \frac{(x^2+y^4)y^2 - xy^2(2x)}{(x^2+y^4)^2} = \frac{(y^4-x^2)y^2}{(x^2+y^4)^2}$; when $x = 0$, we need to use the definition. $\frac{\partial f}{\partial x}(0,y) = \lim_{t\to 0} \frac{f((0,y)+t(1,0)) - f(0,y)}{t} = \lim_{t\to 0} \frac{f(t,y)-0}{t} = \lim_{t \to 0} \frac{ty^2}{t(t^2+y^4)} = \frac{1}{y^2}$, which is defined if and only if $y \neq 0$. Note that $\lim_{x\to 0} \frac{(y^4-x^2)y^2}{(x^2+y^4)^2} = \frac{1}{y^2}$, so we have $\frac{\partial f}{\partial x}(x,y) = \frac{(y^4-x^2)y^2}{(x^2+y^4)^2}$ if $(x,y) \neq (0,0)$. $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = 0$, so

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{(y^4-x^2)y^2}{(x^2+y^4)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can do the same for $\frac{\partial f}{\partial y}$. Now, partial derivatives exist but are not continuous, so it is inconclusive for the existence of $T_{(0,0)}f$. However, if it exists, $T_{(0,0)}f = (\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = (0,0)$, but

$$\lim_{(s,t)\to(0,0)} \frac{|f((0,0)+(s,t)) - f(0,0) - (0,0)\cdot(s,t)|}{\|(s,t)\|_2} = \lim_{(s,t)\to(0,0)} \frac{st^2}{(s^2+t^4)\sqrt{s^2+t^2}}$$

does not exist, so it is not equal to 0. Therefore, the conclusion is f is not differentiable at $(0,0)$. ☐

Example 2. (8.14.4) A differentiable scalar field f has, at the point $(1,2)$, directional derivatives $+2$ in the direction toward $(2,2)$ and -2 in the direction toward $(1,1)$. Determine the gradient vector at $(1,2)$ and compute the directional derivative in the direction toward $(4,6)$.

Solution. Let the gradient vector $T_{(1,2)}f = (h,k)$. Then $2 = T_{(1,2)}f((2,2)-(1,2)) = (h,k)\cdot(1,0) = h$, and $-2 = T_{(1,2)}f((1,1)-(1,2)) = (h,k)\cdot(0,-1) = -k$, so $(h,k) = (2,2)$. The directional derivative toward $(4,6)$ is $(2,2)\cdot((4,6)-(1,2)) = (2,2)\cdot(3,4) = 14$. ☐

Example 3. Let $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z^2$. Identify $\mathbb{C}$ with $\mathbb{R}^2$ and write f as $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution. Let $z = x + yi$. $z^2 = (x+yi)^2 = (x^2-y^2) + 2xyi$. Therefore, $f(x,y) = (x^2 - y^2, 2xy)$. ☐