MA 1C (SECTION 11) RECITATION 10

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1. Trigonometric Identities

- $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
- $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$
- $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sin x\cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$
- $\cos x\sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$
- $\cos x\cos y = \frac{1}{2}(\cos(x+y) + \sin(x-y))$
- $\sin x\sin y = -\frac{1}{2}(\cos(x+y) - \cos(x-y))$

2. Examples

Example 1. Evaluate $\iint_S x+y\,dxdy$, where $S = \{(x,y) : |x|+|y| \le 1\}$.

Solution. Note that $x+y$ is a polynomial function and is continuous, and S is a type I region, we can apply Fubini's Theorem to get

$$\begin{aligned}\iint_S x+y\,dxdy &= \int_{-1}^0\int_{-1-x}^{1+x} x+y\,dydx + \int_0^1\int_{-1+x}^{1-x} x+y\,dydx\\ &= \int_{-1}^0 \left[xy + \frac{y^2}{2}\right]_{-1-x}^{1+x} dx + \int_0^1 \left[xy + \frac{y^2}{2}\right]_{-1+x}^{1-x} dx\\ &= \int_{-1}^0 2x(1+x)\,dx + \int_0^1 2x(1-x)\,dx\\ &= \left[x^2 + \frac{2x^3}{3}\right]_{-1}^0 + \left[x^2 - \frac{2x^3}{3}\right]_0^1\\ &= 0 - (1-\tfrac{2}{3}) + (1-\tfrac{2}{3}) - 0\\ &= 0.\end{aligned}$$

□

Example 2. Evaluate the line integral $\int_C f\cdot d\alpha$, where $f(x,y) = (x+y)\mathbf{i} + (x+y)\mathbf{j}$, and C is the path from $(-1,1)$ to $(1,1)$ along the parabola $y = x^2$.

Solution. Let $\alpha : [-1,1] \to \mathbb{R}^2$ be defined as $\alpha(t) = (t,t^2)$. Then

$$\begin{aligned}\int_C f\cdot d\alpha &= \int_{-1}^1 f(\alpha(t))\cdot\alpha'(t)\,dt\\ &= \int_{-1}^1 (t+t^2, t+t^2)\cdot(1,2t)\,dt\\ &= \int_{-1}^1 t + t^2 + 2t^2 + 2t^3\,dt\\ &= \left[\frac{t^2}{2} + t^3 + \frac{t^4}{2}\right]_{-1}^1\\ &= 2.\end{aligned}$$

Another way is to notice that $f(x,y) = \nabla\phi(x,y)$, where $\phi(x,y) = \frac{1}{2}(x+y)^2$, so the line integral is independent of the path. By the second fundamental theorem of calculus for line integrals (Theorem 10.3), $\int_C f\cdot d\alpha = \int_C \nabla\phi\cdot d\alpha = \phi(1,1) - \phi(-1,1) = 2$. □

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Example 3. Find the area of the region bounded by the curve $\alpha(t) = \sin 2t\mathbf{i} + \sin t\mathbf{j}$ for $t \in [0, \pi]$.

Solution. Let C be the curve and let R be the region bounded by the curve, and let $\alpha(t) = (x(t), y(t))$. We can apply Green's theorem to find the area of this region.

$$\begin{aligned}
\text{Area } &= \int_R 1\, dxdy \\
&= \tfrac{1}{2}\int_C xdy - ydx \\
&= \tfrac{1}{2}\int_0^\pi x(t)y'(t) - y(t)x'(t)\, dt \\
&= \tfrac{1}{2}\int_0^\pi \sin 2t \cos t - 2\sin t \cos 2t\, dt \\
&= \tfrac{1}{2}\int_0^\pi \tfrac{1}{2}(\sin 3t + \sin t) - (\sin 3t + \sin(-t))\, dt \\
&= \tfrac{1}{4}\int_0^\pi - \sin 3t + 3\sin t\, dt \\
&= \tfrac{1}{4}\left[\tfrac{\cos 3t}{3} - 3\cos t\right]_0^\pi \\
&= \tfrac{1}{4}(-\tfrac{1}{3} + 3 - (\tfrac{1}{3} - 3)) \\
&= \tfrac{4}{3}.
\end{aligned}$$

□

Example 4. Evaluate $\iiint_S \sqrt{x^2 + y^2}\, dxdydz$, where S is the solid formed by the upper nappe of the cone $z^2 = x^2 + y^2$ and the plane $z = 1$.

Solution. Note that $\sqrt{x^2 + y^2}$ is continuous, and the solid S is bounded, by Fubini's Theorem and cylindrical coordinates,

$$\begin{aligned}
\iiint_S \sqrt{x^2 + y^2}\, dxdydz &= \int_0^1 \int_0^{2\pi} \int_r^1 r^2\, dzd\theta dr \\
&= 2\pi \int_0^1 r^2(1 - r)\, dr \\
&= 2\pi \left[\tfrac{r^3}{3} - \tfrac{r^4}{4}\right]_0^1 \\
&= \tfrac{\pi}{6}.
\end{aligned}$$

□

Example 5. Let $F(x, y, z) = -z\mathbf{i} + xy\mathbf{k}$. Does there exist a continuously differentiable vector field G such that $F = \nabla \times G$ in $\mathbb{R}^3$? If so, find such a G.

Solution. Since the divergence $\nabla \cdot F = 0$, by Theorem 12.5, there exists a vector field G such that $F = \nabla \times G$. Now, we want to find $L(x, y, z)$, $M(x, y, z)$ and $N(x, y, z)$ such that

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = -z,\ \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} = 0, \text{ and } \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = xy.$$

Set $N = 0$. Then we have $\frac{\partial M}{\partial z} = z$ and $\frac{\partial L}{\partial z} = 0$, or $M(x, y, z) = \frac{z^2}{2} + m(x, y)$ and $L(x, y, z) = \ell(x, y)$. This gives $\frac{\partial m}{\partial x} - \frac{\partial \ell}{\partial y} = xy$. We can take $\ell = 0$, and $m = \frac{x^2 y}{2}$. Therefore, a choice of $G(x, y, z)$ is $(0, \frac{x^2 y + z^2}{2}, 0)$. □