MA 1C (SECTION 11) RECITATION 1

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1. Introduction

Here are some basic information about the TA:

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2. Basic topology in $\mathbb{R}^n$

Let $x_1, x_2, \ldots, x_j, \ldots \in \mathbb{R}^n$. We say that this sequence is converging to $x \in \mathbb{R}^n$ if:
• for all $\epsilon > 0$, there exists $J \in \mathbb{N}$ such that for all $j \geq J$, $\|x_j - x\| < \epsilon$, or
• each component of (x_j) forms a convergent sequence in $\mathbb{R}$.

Let $S \subseteq \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is

- **interior point** of S if:
  there exists $r > 0$ such that $B_x(r) \subseteq S$;
- **boundary point** of S if:
  for all $r > 0$, $B_x(r) \cap S \neq \emptyset$ and $B_x(r) \cap (\mathbb{R}^n \backslash S) \neq \emptyset$, or
  there exists a sequence of points $s_1, s_2, \ldots, s_j, \ldots \in S$ such that $\lim_{j \to \infty} s_j = x$;
- **exterior point** of S if:
  there exists $r > 0$ such that $B_x(r) \subseteq \mathbb{R}^n \backslash S$, i.e. x is an interior point of $\mathbb{R}^n \backslash S$.

The collection of interior points of S, boundary points of S and exterior points of S are denoted as $\text{int}(S)$, ∂S and $\text{ext}(S)$ respectively. Note that $\text{int}(S)$, ∂S and $\text{ext}(S)$ are mutually disjoint, and $\text{int}(S) \cup \partial S \cup \text{ext}(S) = \mathbb{R}^n$.

S is **open** if:
• $S = \text{int}(S)$, i.e. for all $x \in S$, there exists $r > 0$ such that $B_x(r) \subseteq S$.
S is **closed** if:
• $\partial S \subseteq S$, i.e. for all convergent sequences $s_1, s_2, \ldots, s_j, \ldots$ such that $\lim_{j \to \infty} s_j = x$, $x \in S$, or
• $\mathbb{R}^n \backslash S$ is open.
(The first condition is more for disproving a set S being closed.)

An arbitrary union of open sets and a finite intersection of open sets are open; a finite union of closed sets and an arbitrary intersection of closed sets are closed.

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The **closure** of S is $\bar{S}$, defined as:
• $\bar{S} = S \cup \partial S$, or
• $\bar{S} = \bigcap\{T \subseteq \mathbb{R}^n : T \text{ is closed and } S \subseteq T\}$.
The second condition means $\bar{S}$ is the smallest closed set containing S, i.e. for all closed T such that $S \subseteq T$, we also have $\bar{S} \subseteq T$.

S is **compact** if:
• for an arbitrary open covering of S, there is a finite subcover, or
• for a countable open covering of S, there is a finite subcover.
This is because every open ball, and hence open set, is the union of open balls with rational centers and rational radii.

Theorem 1 (Heine-Borel)**.** *$S \subseteq \mathbb{R}^n$ if and only if S is closed and bounded.*

In the proof, Prof. Kechris used the fact that for nested intervals $I_1, I_2, \ldots$ such that the lengths go to 0, $\bigcap I_j$ is a singleton $\{x\}$. In fact, if $I_j = [a_j, b_j]$, then we can show that $x = \lim\limits_{j\to\infty} a_j = \lim\limits_{j\to\infty} b_j$ is inside all I_j. In $\mathbb{R}^n$, we have the same statement by using closed boxes C_j instead of closed intervals I_j. In fact, we can further generalize it.

Theorem 2 (Cantor's intersection)**.** *If $C_1, C_2, \ldots$ is a sequence of nested, non-empty, closed and bounded subsets of $\mathbb{R}^n$, then $\bigcap C_j \neq \emptyset$.*

3. Limit and Continuity

Two ways to view $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$:
• a vector field, or
• a mapping.

Roughly speaking, the **limit of a function** f at $x \in D$ is ℓ if the function value $f(y)$ gets close to ℓ when y gets close to x. Formally, $\lim\limits_{y\to x} f(y) = \ell$ if:
for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in D$ satisfying $0 < \|y - x\| < \delta$, we have $\|f(y) - \ell\| < \epsilon$.

Roughly speaking, f is **continuous** if anything close together in D is mapped to something close together in $\mathbb{R}^m$. Formally, f is continuous at $x \in D$ if:
$\lim\limits_{y\to x} f(y) = f(x)$, i.e. for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in D$ satisfying $\|y - x\| < \delta$, we have $\|f(y) - f(x)\| < \epsilon$.

Conversely, f is **discontinuous** at $x \in D$ if:
• there exists a sequence $x_1, \ldots, x_j, \ldots \in D$ such that $\lim\limits_{j\to\infty} x_j = x$, but $\lim\limits_{j\to\infty} f(x_j) \neq f(x)$, or
• there exist two sequences $x_1, \ldots, x_j, \ldots \in D$ and $x'_1, \ldots, x'_j, \ldots \in D$ such that $\lim\limits_{j\to\infty} x_j = \lim\limits_{j\to\infty} x'_j = x$, but $\lim\limits_{j\to\infty} f(x_j) \neq \lim\limits_{j\to\infty} f(x'_j)$.

4. Examples

Example 1. Show that $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ is open.

Solution. In *geometric argument*, we see that for every point $(x, y, z) \in S$, it is of distance $r > 0$ from the closest point on the surface $x^2 + y^2 = 1$. So $B_{(x,y,z)}(r) \subseteq S$, meaning that S is open.

In *rigorous argument*, for every point $(x, y, z) \in S$, let $r = 1 - \sqrt{x^2 + y^2}$. Now, for every point $(x', y', z') \in B_{(x,y,z)}(r)$, $\sqrt{x'^2 + y'^2} \leq \sqrt{(x' - x)^2 + (y' - y)^2} + \sqrt{x^2 + y^2}$ by triangle inequality, which is less than or equal to $r + \sqrt{x^2 + y^2} < 1$, meaning that $(x', y', z') \in S$. Hence, $B_{(x,y,z)}(r) \subseteq S$. Therefore, S is open. $\square$

Example 2. Find the interior and boundary of the set of all irrational numbers $S = \mathbb{R}\backslash\mathbb{Q}$.

Solution. For all $x \in S$, for all $r > 0$, $B_x(r)$ contains both rational and irrational points, so $x \notin \text{int}(S)$. In other words, $\text{int}(S) = \emptyset$.

For all $x \in \mathbb{R}$, for all $r > 0$, $B_x(r)$ contains both rational and irrational points, so $x \in \partial S$. In other words, $\partial S = \mathbb{R}$.
(Hence, S is neither open nor closed since $S \neq \text{int}(S)$ and $\partial S \nsubseteq S$.) $\square$

Example 3. Find all $(x, y, z) \in \mathbb{R}^3$ such that $f(x, y, z) = \frac{e^{x^2+yz}}{x-y}$ is continuous.

Solution. $g(x, y, z) = x^2 + yz$ and $x - y$ are polynomials and are continuous, and $h(x) = e^x$ is also continuous, so the composite function $h \circ g(x, y, z) = e^{x^2+yz}$ is continuous. A rational function is continuous at all points except those such that the denominator is 0, so $\frac{e^{x^2+yz}}{x-y}$ is continuous at $\mathbb{R}^3\backslash\{(x, y, z) : x = y\}$.

Since $f(x, y, z)$ are undefined on $\{(x, y, z) : x = y\}$, it is not continuous at these points. Therefore, all the points such that $f(x, y, z)$ is continuous at is $\mathbb{R}^3\backslash\{(x, y, z) : x = y\}$. $\square$