# MA 1C CALCULUS OF ONE AND SEVERAL VARIABLES AND LINEAR ALGEBRA, ANALYTICAL TRACK

LECTURE 15

## 1. Examples and Applications

### 1.1. Volume below the graph of a function.

Let $B = [a_1, b_1] \times [a_2, b_2]$, and let $f : B \to [0, \infty)$ be a continuous function. Let $C = \{(x, y, z) : (x, y) \in B \text{ and } 0 \le z \le f(x, y)\}$.

We have defined $\mathrm{vol}(C) = \iiint\limits_C 1\, dx\, dy\, dz$. We will now see (as in the case of one variable) that $\mathrm{vol}(C) = \iint\limits_B f(x, y)\, dx\, dy$, which is the "volume of the solid below the graph of $f$".

Indeed, by Fubini's theorem,

$$\iiint\limits_C 1\, dx\, dy\, dz = \int_{a_1}^{b_1} \Big( \int_{a_2}^{b_2} \Big( \int_0^{f(x,y)} 1\, dz \Big) dy \Big) dx = \int_{a_1}^{b_1} \Big( \int_{a_2}^{b_2} f(x, y)\, dy \Big) dx = \iint\limits_B f(x, y)\, dx\, dy.$$

Note that $A(x) := \int_{a_2}^{b_2} f(x, y)\, dy$ for $x \in [a_1, b_1]$ is the area of a slice of $C$ by a plane parallel to the $(y, z)$-plane, so $\mathrm{vol}(C)$ is the integral of the area of these slices.

### 1.2. Volumne of the ellipsoid.

Let $E = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$, where $a, b, c > 0$.

By applying Fubini's theorem,

$$\begin{aligned}
\mathrm{vol}(E) &= \iiint\limits_E 1\, dx\, dy\, dz \\
&= \int_{-a}^{a} \bigg( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} \bigg( \int_{-c\sqrt{1-(\frac{x}{a})^2-(\frac{y}{b})^2}}^{c\sqrt{1-(\frac{x}{a})^2-(\frac{y}{b})^2}} 1\, dz \bigg) dy \bigg) dx \\
&= \int_{-a}^{a} \bigg( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} 2c\sqrt{1 - (\tfrac{x}{a})^2 - (\tfrac{y}{b})^2}\, dy \bigg) dx \\
&= \int_{-a}^{a} \pi bc \Big( 1 - \big(\tfrac{x}{a}\big)^2 \Big)\, dx \qquad \text{(by substitution of sin function)} \\
&= \pi bc \Big[ x - \frac{x^3}{3a^2} \Big]_{-a}^{a} = \frac{4}{3}\pi abc.
\end{aligned}$$

In particular, the volume of the unit ball is $\frac{4\pi}{3}$ since $a = b = c = 1$.

### 1.3. Volume of the tetrahedron.

Let $T = \{(x, y, z) : x, y, z \ge 0, x + y + z \le 1\}$. This is the solid under the function $f(x, y) = 1 - x - y$, where $f$ is defined on the triangle $\{(x, y) : x, y \ge 0, x + y \le 1\}$.

By Section 1.1,

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$$\begin{aligned}
\text{vol}(T) &= \iint_T 1 - x - y \; dx\, dy \\
&= \int_0^1 \left( \int_0^{1-x} \; 1 - x - y \; dy \right) dx \\
&= \int_0^1 \left[ y - xy - \tfrac{y^2}{2} \right]_0^{1-x} \; dx \\
&= \int_0^1 \; (1-x) - x(1-x) - \tfrac{(1-x)^2}{2} \; dx \\
&= \int_0^1 \; \tfrac{(x-1)^2}{2} \; dx \\
&= \left[ \tfrac{(x-1)^3}{6} \right]_0^1 = \tfrac{1}{6}.
\end{aligned}$$

1.4. **Pappus' Theorem.** (c. 290-350 AD)

Let $\psi, \phi : [a,b] \to [0,\infty)$ such that $\psi \geq \phi$, i.e. $\psi(x) \geq \phi(x)$ for all $x \in [a,b]$. We can then rotate the region bounded by $\psi$, $\phi$, $x = a$, and $x = b$ about the $x$-axis to form a solid of revolution $S$. We are going to calculate the volume of $S$.

Consider the solid of revolution where we rotate $\{(x,y) : a \leq x \leq b \text{ and } y \leq \psi(x)\}$; call it $S_1$. Similarly, the solid of revolution for $\{(x,y) : a \leq x \leq b \text{ and } y \leq \phi(x)\}$ is called $S_2$. Then $\text{vol}(S) = \text{vol}(S_1) - \text{vol}(S_2)$ (note that the boundaries can be ignored since the surface of revolution of $\phi$ has content 0).

Now, $S_1 = \{(x,y,z) : a \leq x \leq b, -\psi(x) \leq y \leq \psi(x), -\sqrt{\psi(x)^2 - y^2} \leq z \leq \sqrt{\psi(x)^2 - y^2}\}$, since for each $x$, the slice of $S_1$ is a disc in the $(y,z)$-plane of radius $\psi(x)$.

$$\begin{aligned}
\text{vol}(S_1) &= \int_a^b \left( \int_{-\psi(x)}^{\psi(x)} \left( \int_{-\sqrt{\psi(x)^2-y^2}}^{\sqrt{\psi(x)^2-y^2}} \; 1 \; dz \right) dy \right) dx \\
&= \int_a^b \left( \int_{-\psi(x)}^{\psi(x)} \; 2\sqrt{\psi(x)^2 - y^2} \; dy \right) dx \\
&= \int_a^b \; \pi \psi(x)^2 \; dx.
\end{aligned}$$

Geometrically, for each $x \in [a,b]$, $\pi\psi(x)^2$ is the area of the disc with radius $\psi(x)$, and $\text{vol}(S_1)$ is the integral of these areas, according to Section 1.1.

Similarly, for $S_2$, $\text{vol}(S_2) = \int_a^b \; \pi\phi(x)^2 \; dx$. Hence, $\text{vol}(S) = \pi \int_a^b \; \psi(x)^2 - \phi(x)^2 \; dx$.

**Definition.** The *centroid* of a Jordan measurable bounded set $Q \subseteq \mathbb{R}^2$ is the point

$$(\overline{x}_Q, \overline{y}_Q) = \left( \frac{\iint_Q \; x \; dx\, dy}{\text{area}(Q)}, \frac{\iint_Q \; y \; dx\, dy}{\text{area}(Q)} \right).$$

Note that $\text{area}(Q) = \iint_Q \; 1 \; dx\, dy$, which is the 2-dimensional volume. In fact, the first coordinate gives the average distance from $(x,y) \in Q$ to $y$-axis, and the second coordinate gives the average distance from $(x,y) \in Q$ to $x$-axis. The centroid is the center of mass of $Q$ if it has density 1.

In our case of type I region, $\overline{y} = \frac{\int_a^b \left( \int_{\phi(x)}^{\psi(x)} \; y \; dy \right) dx}{A} = \frac{\int_a^b \; \psi(x)^2 - \phi(x)^2 \; dx}{2A}$, where $A$ is the area of the region.

Therefore, $\text{vol}(S) = 2\pi\overline{y}A = 2\pi \times$ (average distance of $Q$ from axis of revolution) $\times$ area.

**Example.** The volume of a torus, obtained by revolving a circle centered at $(a,b)$ and radius $r$ about $x$-axis can be given by $2\pi b(\pi r^2) = 2\pi^2 br^2$, since it is obvious that $\overline{y} = b$.