MA 17 — HOW TO SOLVE IT
LECTURE 9

TONY WING HONG WONG

1. Probability

1.1. Theory.

Probability problems are usually quite deep. Apart from probability theory, like independence, expectation etc, you may also need combinatorics, analysis and many other tools.

• Expectation

If X is a discrete random variable taking values $\{x_1, x_2, \dots\}$, then

$$\mathbb{E}[X] = \sum_i x_i \mathbb{P}(X = x_i).$$

If X is a continuous random variable with density function $f(x)$, then

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

(Recall that density function $f(x)$ is such that $\mathbb{P}(X \leq x_0) = \int_{-\infty}^{x_0} f(x)dx$.)

Let X and Y be random variables. Then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. If X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

• Relationship with Area

Sometimes it is convenient to consider the probability space as a geometric object, and calculate the probability by find out the area.

1.2. Problems.

(2002 B1) (*) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

(2001 A2) You have coins $C_1, C_2, \dots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of fallings heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answers as a rational function of n.

(1993 B2) (*) Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

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(1992 A6) Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

(1993 B3) Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

(1989 B1) (*) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

(1985 B4) (*) Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

2. Inequalities

2.1. Theory.

• AM-GM Inequality

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Then

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

where equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

• Power Mean Inequality

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let $p \geq q$. Then

$$\left(\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\right)^{1/p} \geq \left(\frac{a_1^q+a_2^q+\cdots+a_n^q}{n}\right)^{1/q},$$

where equality holds if and only if $a_1 = a_2 = \cdots = a_n$. The AM-GM inequality is a special case with $p = 1$ and $q = 0$.

• Cauchy-Schwarz Inequality

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be real numbers. Then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2,$$

where equality holds if and only if there exists an "extended" real number k such that $a_i = kb_i$ for all $i = 1, 2, \ldots, n$. In particular, $n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2$.

There are a few more famous inequalities, including the rearrangement inequality, majorization inequality, Jensen Inequality, Muirhead inequality etc. Many other inequalities arise from analysis, for example, $e^x > 1 + x$ for all real x, or the Sterling's formula:

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e n^{n+\frac{1}{2}} e^{-n}.$$

2.2. **Problems.**

(2005 B2) Find all positive integers $n, k_1, k_2, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

(2004 B2) (*) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

(2003 A2) (*) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \ldots a_n)^{1/n} + (b_1 b_2 \ldots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n)]^{1/n}.$$

(1996 B2) (*) Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(1988 B2) (*) Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

3. Homework

You can work on those problems marked with an asterisk (*), but you do not need to submit your work for this week.