MA 17 — HOW TO SOLVE IT
LECTURE 8

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1. Geometry

1.1. Theory.

Most geometry problems are of high-school level, so if you are good at geometry in IMO level, then you probably want to try on these problems in the Putnam competition.

• Pure Geometry

Euler's theorem for polyhedrons: $V - E + F = 2$, where V is the number of vertices, E the number of edges and F the number of faces.

Menelaus theorem: Let ℓ be a line that cuts through ΔABC at D on AB, E on AC and F on the extension of BC. Then

$$\frac{AD}{DB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = 1.$$

Four centers of triangle: The circumcenter is the intersection of perpendicular bisectors, incenter the intersection of angle bisectors, centroid the intersection of medians and orthocenter the intersection of altitudes. The centroid always partitions the median at ratio $2 : 1$, and circumcenter, centroid and orthocenter always form a straight line.

Area of triangle: Let ΔABC have side lengths $|AB| = c$, $|AC| = b$, $|BC| = a$. Let R be the circumradius of ΔABC, r be the inradius and $s = \frac{a+b+c}{2}$. Then the area of ΔABC is

$$\frac{1}{2}ab\sin\angle C = \frac{abc}{4R} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

There are many other theorems that you need to be familiar with, including sine law and cosine law etc.

• Coordinate Geometry

Area of polygons: Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n vertices of an n-sided polygon in $\mathbb{R}^2$, arranged counter-clockwisely. Then the area of this polygon is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{vmatrix}.$$

Surface of revolution: Let $y = f(x)$ be a differentiable function which is positive on the interval $[a, b]$. Then the area of the surface of revolution about x-axis is

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$

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• Trigonometry

There are a lot of identities in trigonometry:

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$

1.2. **Problems.**

(2010 B2) Given that A, B, and C are noncolinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

(2008 B1) (*) What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

(2002 A2) (*) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

(2002 B2) Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

(2001 A4) (*) Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

(2000 A3) (*) The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

(2000 A5) (*) Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

(1999 B1) (*) Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chose on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

(1998 A1) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

(1998 A5) (*) Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{n} 3D_j.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

(1998 A6) Let A, B, C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

(1998 B3) Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.

(1997 A1) (*) A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

(1994 A3) (*) Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

(1993 B5) Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

2. Trigonometry

(2009 A3) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

(2003 A3) Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

3. Special Problem in Number Thoery

1. (†) Let a and b be positive integers such that $a \geq b$. Show that for all prime $p \geq 5$,

$$\tbinom{pa}{pb} \equiv \tbinom{a}{b} \pmod{p^3}.$$

(*Hint*: You may want to first show the special case $\binom{2p}{p} \equiv 2 \pmod{p^3}$ for all primes $p \geq 5$.)

4. Homework

Please submit your work on three of the problems that are marked with an asterisk (*). If you can finish one of the problems marked with (†), then submitting the complete solution to that problem is sufficient for your homework, and you do not have to work on other problems.