MA 17 — HOW TO SOLVE IT
LECTURE 7

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1. Number Theory

1.1. **Theory.**

Most number theory problems are related to even-odd parity, divisibility, primes, perfect squares etc. The variety of problems is great, so there is no fixed pattern in solving these problems. The general advice is the same as always: keep an open mind, understand all the conditions and your goal, keep trying different methods and have faith that you will stumble onto a simple solution.

• Congruence

Very often, considering the right congruence will simply the problem significantly. In particular, it is good to bear in mind that $n^2 \equiv 1 \pmod 4$ and $n^2 \equiv 1 \pmod 8$ if $2 \nmid n$, $n^2 \equiv 1 \pmod 3$ if $3 \nmid n$, $n^2 \equiv 1 \pmod 6$ if $\gcd(n, 6) = 1$.

• Many random facts

Whenever you see $\underbrace{111\dots1}_{n\text{ 1's}}$, you can consider $\frac{10^n-1}{9}$. If n is even, you may also consider $\underbrace{111\dots1}_{\frac{n}{2}\text{ 1's}}\times 1\underbrace{00\dots0}_{\frac{n}{2}-1\text{ 0's}}1$. When you have $\frac{h}{a} + \frac{k}{b} = 1$ or $ab - ha - kb = 0$, you may consider $(a-k)(b-h) = hk$. Fermat's little theorem and Euler's ϕ theorem are nice theorems related to primes (though I have never used them in any Putnam problems so far).

This kind of knowledge only builds up when you work on a lot of problems and try to remember everything you have come across.

1.2. **Problems.**

(2010 A4) Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

(2009 A4) Let S be a set of rational numbers such that
(a) $0 \in S$;
(b) If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$; and
(c) If $x \in S$ and $x \notin \{0, 1\}$, then $1/(x(x-1)) \in S$.
Must S contain all rational numbers?

(2005 A1) (*) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For

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example, $23 = 9 + 8 + 6$.)

(2001 A5) (*) Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

(2001 B4) (*) Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

(2000 A2) (*) Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

(2000 B2) (*) Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

(1998 A4) (*) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} for left to right. For example $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

(1998 B5) Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

(1997 A5) Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

(1993 B5) Show there do not exist four points in the Euclidena plane such that the pairwise distances between the points are all odd integers.

(1992 A3) (*) For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy

$$(x^2 + y^2)^m = (xy)^n.$$

(1989 A1) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

2. Homework

Please submit your work on three of the problems that are marked with an asterisk (*).