MA 17 — HOW TO SOLVE IT
LECTURE 6

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1. Linear Algebra

1.1. **Theory.**

• Jordan normal form/Jordan canonical form

For all square matrix A, there exists a complex matrix P such that $P^{-1}AP = J$ is a Jordan normal form. The Jordan normal form encodes a lot of important information of A, including the algebraic and geometric multiplicities of its eigenvalues. Jordan normal forms can be useful when dealing with nilpotent matrices. For example, if $A^k = 0$, then all eigenvalues are 0 and all the Jordan blocks are of size $\leq k$. In particular, it shows that $k \leq n$.

• p-rank of an integer matrix $A \leq$ rank of A over $\mathbb{R}$

The p-rank of A is the number of nonzero rows in the row reduced echelon form over the finite field $\mathbb{Z}_p$, i.e. row reductions are done mod p. It is also useful to bear in mind that

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{n\times n}$$

has 2-rank n if n is even, 2-rank $n-1$ if n is odd, and rank n over $\mathbb{R}$.

• Special matrices like Hadamard matrix and Vandermonde matrix

Hadamard matrix is a square matrix with entries ± 1 such that all the rows are mutually orthogonal over $\mathbb{R}$. It has a lot of deep properties in analysis and combinatorics. Vandermonde matrix is

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{pmatrix}_{n\times n}$$

which has determinant $\prod_{i<j}(a_j - a_i)$.

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1.2. **Problems.**
1. There are 23 soccer players (I would rather say football players), each with a body mass a positive real number. A player is a "good choice of referee" if the other 22 players can be separated into two teams of 11 players such that these two teams have the same total body masses. If each of the 23 players is a good choice of referee, show that all players have the same body mass.

(2011 A4) For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

(2011 B4) (†) In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.
Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

(2008 A2) (*) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entires are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

(2006 B4) Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace of V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum of $Z(V)$ over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k.

(2005 A4) Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

(2003 B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

(2002 A4) (*) In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

(1994 A4) (*) Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

(1992 B5) (*) Let D_n denote the value of the $(n-1)\times(n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n\geq 2}$ bounded?

(1991 A2) (*) Let A and B be different $n\times n$ matrices with real entries. If $A^3=B^3$ and $A^2B=B^2A$, can A^2+B^2 be invertible?

(1990 A5) If A and B are square matrices of the same size such that $ABAB=0$, does it follow that $BABA=0$?

(1988 B5) (*) For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n.

One may note that

$$M_1=\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \text{ and } M_2=\begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

(1986 A6) (†) Let $a_1,a_2,\ldots,a_n$ be real numbers, and let $b_1,b_2,\ldots,b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x)=1+\sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1,b_2,\ldots,b_n$ and n (but independent of $a_1,a_2,\ldots,a_n$).

2. Homework

Please submit your work on three of the problems that are marked with an asterisk (*). If you can finish one of the problems marked with (†), then submitting the solution to that problem is sufficient for your homework, and you do not have to work on other problems.