MA 17 — HOW TO SOLVE IT
LECTURE 5

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1. Analysis

(2011 B3) (*) Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

(2010 A6) Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

(2010 B5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

(2009 B5) (*) Let $f : (1,\infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2-(f(x))^2}{x^2((f(x))^2+1)} \text{ for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

(2008 A4) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^\infty \frac{1}{f(n)}$ converge?

(2008 B2) (*) Let $F_0(x) = \ln x$. For $n \ge 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}.$$

(2004 B5) Evaluate

$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

(1994 A1) (*) Suppose that a sequence $a_1, a_2, a_3, \dots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^\infty a_n$ diverges.

(1988 A3) Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\csc\frac{1}{n} - 1\right)^x$$

converges.

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(1988 B4) (*) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/n+1}$.

2. Homework

Please submit your work on three of the problems that are marked with an asterisk (*).