MA 17 — HOW TO SOLVE IT
LECTURE 4

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1. Calculus

(2009 A6) Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)dy$, $b = \int_0^1 f(1,y)dy$, $c = \int_0^1 f(x,0)dx$, $d = \int_0^1 f(x,1)dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0) = b - a \text{ and } \frac{\partial f}{\partial y}(x_0,y_0) = d - c.$$

(2007 B2) Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)dx \right| \le \frac{1}{8} \max_{0 \le x \le 1} |f'(x)|.$$

(2006 A1) (*) Find the volume of the region of points (x,y,z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2).$$

(2006 B5) (*) For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)dx$ and $J(x) = \int_0^1 x(f(x))^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

(2005 A5) Evaluate $\int_0^1 \frac{\ln(x+1)}{x^2+1}dx$.

(1985 A5) (*) Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)dx$. For which integers m, $1 \le m \le 10$ is $I_m \ne 0$?

(1985 B2) (*) Define polynomials $f_n(x)$ for $n \ge 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \ge 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \ge 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

2. Differential Equations

(2010 A3) Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

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for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

(2009 A2) Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \tfrac{1}{gh}, \qquad & f(0) &= 1, \\ g' &= fg^2h + \tfrac{4}{fh}, \qquad & g(0) &= 1, \\ h' &= 3fgh^2 + \tfrac{1}{fg}, \qquad & h(0) &= 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

(2005 B5) (*) Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\tfrac{\partial^2}{\partial x_1^2} + \cdots + \tfrac{\partial^2}{\partial x_n^2}\right) P(x_1, \ldots, x_n) = 0 \qquad \text{(identically)}$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

(1988 A2) (*) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

(1987 A3) (*) For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.
(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

3. Homework

Please submit your work on three of the problems that are marked with an asterisk (*), with at least one problem in each section.