MA 17 — HOW TO SOLVE IT
LECTURE 3

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1. Polynomials

(1952 A1) The polynomial $p(x)$ has all integral coefficients. The leading coefficient, the constant term, and $p(1)$ are all odd. Show that $p(x)$ has no rational roots.

(2011 B2) (*) Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

(2010 B4) Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

(2008 B4) Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

(2007 B1) (*) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$. [Note: one must assume f is nonconstant.]

(2007 B4) Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

(2004 B1) (*) Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1}r,\ c_n r^3 + c_{n-1}r^2 + c_{n-2}r,\ \ldots,\ c_n r^n + c_{n-1}r^{n-1} + \cdots + c_1 r$$

are integers.

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2. Functions

(2010 A2) (*) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n)-f(x)}{n}$$

for all real numbers x and all positive integers n.

(2010 B5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

(2008 A1) (*) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

(2005 B3) Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

(2000 B4) Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

3. Homework

Please submit your work on three of the problems that are marked with an asterisk (*), with at least problem in each section.