

MA 17 — HOW TO SOLVE IT

LECTURE 2

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1. AMENDMENT IN COURSE POLICIES

I am very sorry to say that there is a need for a change to the non-compulsory homework policy in this class. I hope this does not upset anyone, since suitable amount of homework can ensure that everyone masters the skills taught during the lectures.

Starting from this week, two to three problems will be assigned as homework each week, and grading will be based on the amount of efforts spent on the homework problems. Individual work is expected, as one of our goals of this course is to train for the Putnam competition.

2. REMARK FROM PREVIOUS WEEK

Last week, we discussed about proof by contradiction and observing patterns/induction. When we do not have any other ideas, it never hurts trying to use these two methods. In particular, it is always a wise move to start tackling a problem by working out a few examples.

3. PIGEON-HOLE PRINCIPLE

1. Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x , and let $\{x\} = x - \lfloor x \rfloor$ be the “fractional part” of x . Prove that for all $\epsilon > 0$ and for all positive irrational number r , there exists a positive integer n such that $\{nr\} < \epsilon$.
2. Show that there exists a positive integer n such that 2^n starts with 2012 in the decimal expression.
3. Let S be a subset of $\{1, 2, \dots, 2012\}$ of size 1007. Show that there exists two distinct elements a and b of S such that $a \mid b$.
4. (*) Show that for any 6 points in a 3×4 rectangle, there exist two of them whose distance is at most $\sqrt{5}$.

(2006 B2) (*) Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

(2000 B6) (†) Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \dots, \pm 1)$ in n -dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

4. COMBINATORICS

(2010 B3) There are 2010 boxes labeled $B_1, B_2, \dots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n . You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution which exactly n balls in each box, regardless of the initial distribution of balls?

(2007 A3) Let k be a positive integer. Suppose that the integers $1, 2, 3, \dots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

(2006 A4) (*) Let $S = \{1, 2, \dots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

- (i) $\pi(k) > \pi(k+1)$ for $k = 1$;
- (ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;
- (iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S , averaging over all permutations of S ?

(2005 A2) (*) Let $\mathbf{S} = \{(a, b) | a = 1, 2, \dots, n, b = 1, 2, 3\}$. A *rook tour* of \mathbf{S} is a polygonal path made up of line segments connecting points p_1, p_2, \dots, p_{3n} in sequence such that

- (i) $p_i \in \mathbf{S}$,
- (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,
- (iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$.

How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

(2003 A1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \dots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$.

5. HOMEWORK

Please submit your work on two of the problems that are marked with an asterisk (*), with one problem in each section. If you can finish the problem marked with (†), then submitting the solution to that problem is sufficient for your homework, and you do not have to work on other problems.