MA 17 — HOW TO SOLVE IT
LECTURE 1

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1. Introduction

Here is some basic information about the instructor:

- Office: 304 Kellogg (connected to Sloan);
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- Webpage: www.math.caltech.edu/∼tonywong/ma17.html
- Office Hour: TBD.

The goal of this course is to raise students' interests in mathematics, enhance their problem solving skills and help them better prepare for the William Lowell Putnam Competition.

In each weekly hour-long lecture, a focus will be chosen, and a selection of problems in the past papers of Putnam related to the focus will be discussed. Sometimes a complete proof will be presented, while at other times, only key ideas are mentioned. In any case, we shall always try to explore how we should start tackling the problems, how far we should push in a seemingly unsuccessful attempt, and how we should write a "precise and concise" proof.

There is no compulsory homework for this course, but practice always makes perfect. Students are highly encouraged to write up the proofs for the problems discussed in class and turn them in to my mailbox. This can help the students practice their presentation skills and digest the materials thoroughly. Students are also encouraged to work on any Putnam problems that interest them and turn them in too.

There are several useful references for this course, and all of them are available online. The first one is the Putnam competition directory, which contains Putnam past papers from 1985–2011 and solutions from 1995–2011. There are also some interesting statistics for the competitions like the scores of the high achievers.

- amc.maa.org/a-activities/a7-problems/putnamindex.shtml

If this website does not provide enough past papers, one may go to the second one, which contains Putnam problems from the first competition in 1938 to the 64th in 2003. Again, this website also contains most of the solutions.

- mks.mff.cuni.cz/kalva/putnam.html

Finally, a website that may prove very useful is the course webpage of the Putnam seminar in Carnegie Mellon University, taught by a former Caltech undergraduate Po-Shen Loh. He is also the deputy team leader of the IMO (International Mathematics Olympiad) USA team.

- www.math.cmu.edu/∼ploh/2012-295.shtml

Date: October 2, 2012.

2. Understanding the Structure of Putnam Competition

There will be two 3-hour sittings on Saturday, December 1, 2012 for the Putnam competition. As your brain needs to work for 6 hours on that day, make sure it has enough energy. A good strategy is to bring snacks on your own. Chocolates, banana or drinks that are rich in glucose are some good choices.

There will be 6 problems in each sitting, numbered 1 to 6. Usually, they are arranged so that problems 1 and 2 require less knowledge in college mathematics. They are more like the mathematical olympiad type questions in high school. Although they require less knowledge, some of them can still be tricky and convoluted. Problems 3 and 4 usually require certain amount of knowledge in freshman to sophomore level of mathematics, but are relatively straight forward. Problems 5 and 6 are usually the hardest ones, requiring a sound knowledge in higher mathematics, or are relatively complicated even the techniques are understandable by a freshman.

Each problem carries 10 points, so the full score of the competition is 120. From the year 2000 to 2011, in order to squeeze into top 100, you need to score around 47 points on average. So if you can finish 5 problems completely, e.g. 3 on the first day and 2 on the second, you are doing really great. However, the actual cutting line varies from year to year, due to the varying level of difficulties of the problems.

In each problem, although theoretically you could get 5 points, it is more likely to be "all or nothing". The distribution of the scores on each problem is always 0, 1, 2 or 8, 9, 10. Hence, it is usually more advisable for you to focus on fewer problems and try to finish them completely than for you to spend equal amount of time on those 6 problems. Besides, it is very important for you to write out the solutions as complete but yet as "precise and concise" as possible, since 1 point could cause you an honorable mention in Putnam.

In order to write out nice solutions, one suggestion is to give the answer to the question in the opening sentence. This saves the graders a lot of time and makes your solutions more pleasant to read. For example, if the question asks "is it always possible to...?", then you should start with your answer "Yes, it is always possible to.../No, it is not always possible to...".

In your solutions, you must define all your variables before using them, and try to make your variables consistent throughout. Do not use the same symbol for two different meanings. Write your proof according to the logical deduction sequence, which is often not in the same order as how you come up with the ideas. Break up long paragraphs into shorter ones to enhance readability. Try to read your whole proof once you finish writing it, and see if you understand the proof by yourself. It also helps if you read all your proofs 15 minutes prior to submission of your answers, since then you can sort of forget you are the author and pretend to read as a third party.

Finally, although we mentioned that the questions are arranged according to the level of difficulty in general, it is very often not the same as the level of difficulty for you. Each person has his/her own strengths, so a problem easy for you may be difficult for the others.

Try to understand your mathematical ability better. Find out which topics you are more familiar with, and practice even more on them. Sometimes, working on your strengths is more efficient than working on your weaknesses.

3. Proof by Contradiction

(2010 B1) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that
$$a_1^m + a_2^m + a_3^m + \cdots = m$$
for every positive integer m?

(2009 A1) Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

(2004 A1) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

4. Observing Patterns/Induction

(2010 A1) Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is at least 3.]

(2006 A2) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

(2006 B3) Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

(2004 A3) Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)