

# Linear Algebra

# Matrix

- **Definition:** A *matrix* is a rectangular array of numbers
- A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix
- The plural of matrix is *matrices*
- A matrix with the same number of rows and columns is called *square*
- Two matrices are equal if they have the same number of rows and columns and the corresponding entries in every position are equal

# Notation

- Let  $m$  and  $n$  be positive integers and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The  $i$ th row of  $A$  is the  $1 \times n$  matrix  $[a_{i1}, a_{i2}, \dots, a_{in}]$
- The  $j$ th column of  $A$  is the  $m \times 1$  matrix:

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

- The  $(i, j)$ th *element* or *entry* of  $A$  is the element  $a_{ij}$
- We can use the notation  $A = [a_{ij}]$  to denote the matrix with its

# Matrix Addition

- **Definition:** Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $m \times n$  matrices; the sum of  $A$  and  $B$ , denoted by  $A + B$ , is the  $m \times n$  matrix that has  $a_{ij} + b_{ij}$  as its  $(i, j)$ th element, that is  $A + B = [a_{ij} + b_{ij}]$
- Note that matrix addition is only defined for matrices of the same size
- **Example:**

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

# Scalar Multiplication

- **Definition:** Let  $k$  be a scalar and  $A$  be a matrix; the *product* of a  $k$  and  $A$ , denoted by  $kA$  is a matrix where every element in  $A$  is multiplied with  $k$
- **Example:**

$$2 \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 8 \\ 4 & 2 & 2 \\ 6 & 2 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

# Matrix Multiplication

- **Definition:** Let  $A$  be an  $m \times k$  matrix and  $B$  be a  $k \times n$  matrix; the *product* of  $A$  and  $B$ , denoted by  $AB$ , is the  $m \times n$  matrix that has its  $(i, j)$ th element equal to the sum of the products of the corresponding elements from the  $i$ th row of  $A$  and the  $j$ th column of  $B$ , that is,  $AB = [c_{ij}]$  where  $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$
- Note that matrix multiplication is not defined when the number of columns in the first matrix is not the same as the number of rows in the second
- **Example:**

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

# Matrix Multiplication is not Commutative

- **Example:** Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- Then

$$AB = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \quad BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

so  $AB \neq BA$

# The Identity Matrix

- **Definition:** The *identity matrix of order  $n$*  is the  $n \times n$  matrix  $I_n = [\delta_{ij}]$  where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- $AI_n = I_m A = A$  when  $A$  is an  $m \times n$  matrix

# Powers of Matrices

■ Powers of matrices can be defined for square matrices

■ When  $A$  is an  $n \times n$  matrix:

$$A^0 = I_n$$

$$A^r = \underbrace{AAA \cdots A}_{r \text{ times}}$$

# Transposes of Matrices

- **Definition:** Let  $A = [a_{ij}]$  be an  $m \times n$  matrix; the transpose of  $A$ , denoted by  $A^T$ , is the  $n \times m$  matrix obtained by interchanging the row and columns of  $A$
- If  $A^T = [b_{ij}]$ , then  $b_{ij} = a_{ji}$  for  $i = 1, 2, \dots, n$
- **Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

# Symmetric Matrices

- **Definition:** A square matrix  $A$  is called *symmetric* if  $A = A^T$
- $A = [a_{ij}]$  is symmetric if  $a_{ij} = a_{ji}$  for  $i$  and  $j$  with  $1 \leq i \leq n$  and  $1 \leq j \leq n$
- **Example:**

$$A = A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

# Inverse of a Matrix

- **Definition:** If  $A$  and  $B$  are  $n \times n$  matrices with  $AB = BA = I$ , the  $B$  is called the *inverse* of  $A$  and  $A$  is said to be invertible. The notation  $B = A^{-1}$  denotes that  $B$  is the inverse of  $A$ .

# Vectors

- A position vector represents a point in  $n$  dimensional space.
- We will use the convention that a vector is an  $n \times 1$  matrix (a column vector).
- Example: a point in 3D space

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

# The Length of a Vector

- The length of a vector  $a$ , denoted as  $\|a\|$ , is defined as:

$$\|a\| = \sqrt{\sum_{i=1}^n a_i^2}$$

or using matrix multiplication:

$$\|a\| = \sqrt{a^T \cdot a}$$

# Vector Dot Product

- The *dot product* of two vectors  $a$  and  $b$  is defined as:
  - $a \cdot b = b \cdot a = \sum_i a_i b_i$
  - $a \cdot b = a^T b$
  - $a \cdot b = \|a\| \|b\| \cos \theta$
- If  $a \cdot b = 0$ , then  $a$  and  $b$  are *orthogonal*