Bayes' Nets

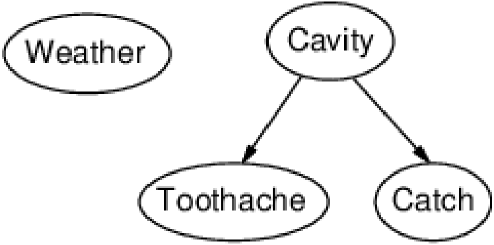
CSC 548, Artificial Intelligence II

Bayesian Networks

- A simple graphical notation for conditional independence assertions and hence for compact specification of full joint distributions
- Syntax:
  - a set of nodes, one per variable
  - a directed acyclic graph (link $\approx$ "directly influences")
  - a conditional distribution for each node given its parents: $P(X_i \mid Parents(X_i))$
- In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

Example

- Topology of network encodes conditional independence assertions:

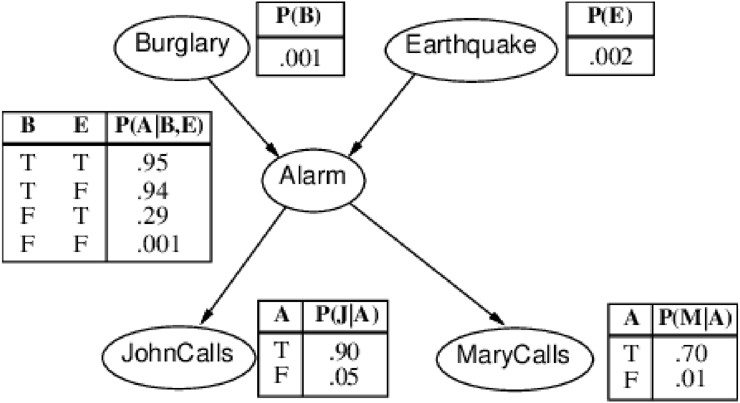


- *Weather* is independent of other variables
- *Toothache* and *Catch* are conditionally independent given *Cavity*

Example

- I am at work, neighbor John calls to say my alarm is ringing, but neighbor Mary does not call. Sometimes it is set off by minor earthquakes. Is there a burglar?
- Variables: $Burglar$, $Earthquake$, $Alarm$, $JonhCalls$, $MaryCalls$
- Network topology reflects "causal" knowledge
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm causes Mary to call
  - The alarm causes John to call

Example Continued



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Compactness

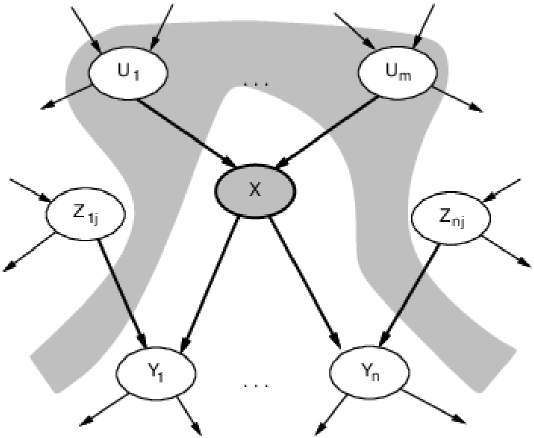
- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values
- Each row requires one number p for $X_i = true$ (the number for $X_i = false$ is just $1 - p$)
- If each variable has no more than k parents, the complete network requires $\mathcal{O}(n \cdot 2^k)$ numbers
- That is, grows linearly with n, vs. $\mathcal{O}(2^n)$ for the full joint distribution
- For the burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

Global Semantics

- "Global" semantics defines the full joint distribution as the product of the local conditional distributions:
  $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(X_i))$
- Example, Burglary net:
  $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$
  $= P(j \mid a)P(m \mid a)P(a \mid \neg b, \neg e)P(\neg b)P(\neg e)$
  $= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998$
  ≈ 0.00063

Local Semantics

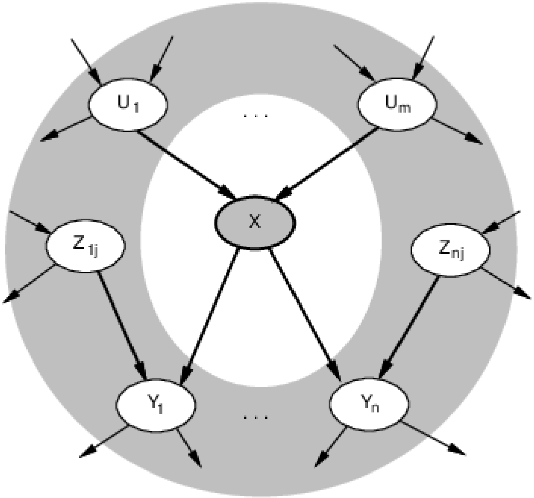
- Local semantics: each node is conditionally independent of its nondescendants given its parent



- Theorem: local semantics $\leftrightarrow$ global semantics

Markov Blanket

- Each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents



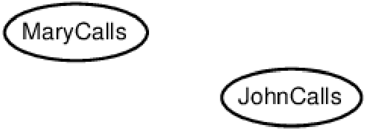
Constructing Bayesian Networks

- Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics
- Choose an ordering of variables $X_1, \ldots, X_n$
- For $i = 1$ to n, add X_i to the network and select parents from $X_1, \ldots, X_{i-1}$ such that
  $P(X_i \mid Parents(X_i)) = P(X_i \mid X_1, \ldots, X_{i-1}$
- This choice of parents guarantees the global semantics:

$$
\begin{aligned}
P(X_1, \ldots, X_n) &= \prod_{i=1}^{n} P(X_i \mid X_1, \ldots, X_{i-1}) \\
&= \prod_{i=1}^{n} P(X_i \mid Parents(X_i)
\end{aligned}
$$

Example

- Suppose we choose the ordering M, J, A, B, E



- $P(J \mid M) = P(J)$?

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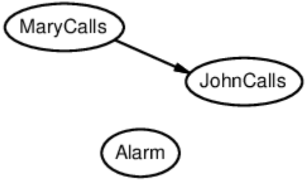
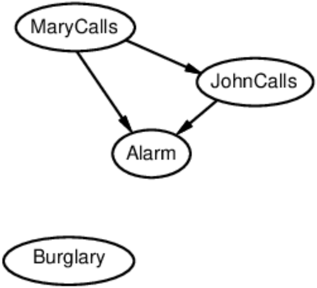


- $P(A \mid J, M) = P(A \mid J)$?
- $P(A \mid J, M) = P(A)$?

Example

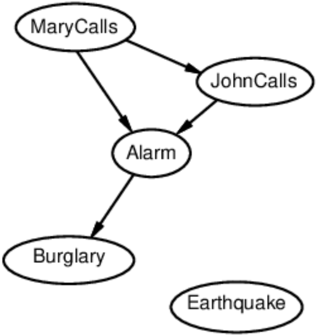
- Suppose we choose the ordering M, J, A, B, E



- $P(B \mid A, J, M) = P(B \mid A)$?
- $P(B \mid A, J, M) = P(B)$?

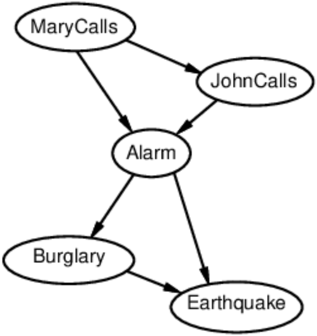
Example

- Suppose we choose the ordering M, J, A, B, E



- $P(E \mid B, A, J, M) = P(E \mid A)$?
- $P(E \mid B, A, J, M) = P(E \mid A, B)$?

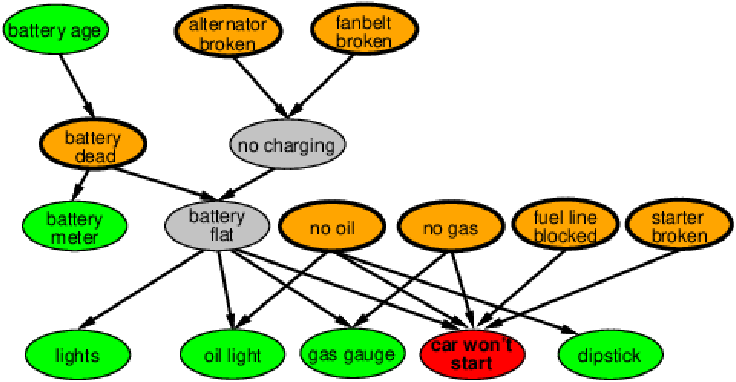
Example



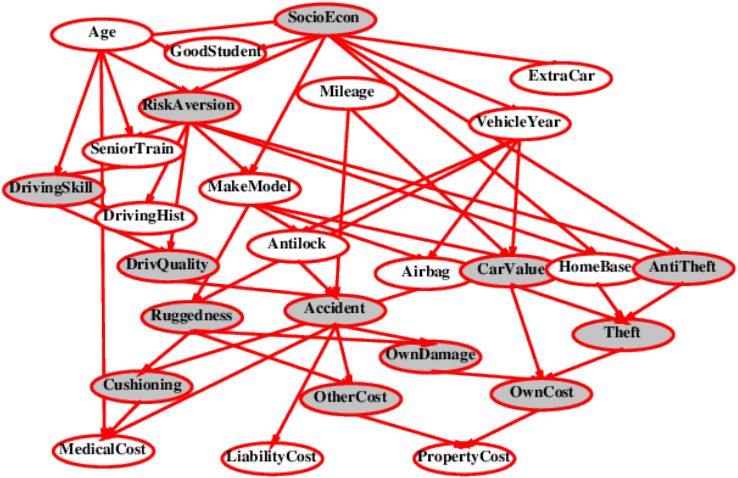
- Deciding conditional independence is hard in noncausal directions
- Assessing conditional probabilities is hard in noncausal directions
- Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed

Example: Car Diagnosis

- Initial evidence: car will not start
- Testable variables (green), "broken, so fix it" variables (orange)
- Hidden variables (gray) ensure sparse structure, reduce parameters



Example: Car Insurance



Compact Conditional Distributions

- CPT grows exponentially with number of parents
- CPT becomes infinite with continuous valued parent or child
- Solution: canonical distributions that are defined compactly
- Deterministic nodes are the simplest case:
  $X = f(Parents(X))$ for some function f
- Examples:
  - $NorthAmerican \leftrightarrow Canadian \vee US \vee Mexican$
  - $\frac{\partial Level}{\partial t} =$ inflow $+$ precipitation - outflow - evaporation

Compact Conditional Distributions

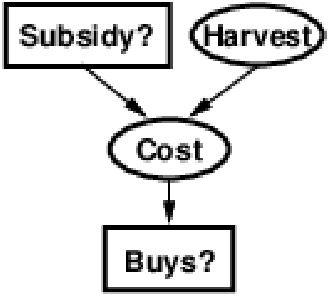
- Noisy-OR distributions model multiple noninteracting causes
  - Parents $U_1, \ldots, U_k$ include all causes
  - Independent failure probability q_i for each cause alone
    $P(X \mid U_1, \ldots, U_j, \neg U_{j+1}, \ldots, \neg U_k) = 1 - \prod_{i=1}^{j} q_i$

Cold	*Flu*	*Malaria*	$P(Fever)$	$P(\neg Fever)$
F	F	F	**0.0**	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.2 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.4	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.12 = 0.6 \times 0.2 \times 0.1$

- Number of parameters linear in number of parents

Hybrid (Discrete + Continuous) Networks

- Discrete (*Subsidy*? and *Buys*?); continuous (*Harvest* and *Cost*)



- Option 1: discretization – possibly large errors large CPTs
- Option 2: finitely parameterized canonical families
- Continuous variable, discrete + continuous parents (e.g. *Cost*)
- Discrete variable, continuous parents (e.g. *Buys*?)

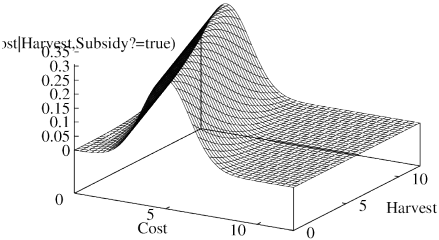
Continuous Child Variables

- Need one conditional density function for child variable given continuous parents, for each possible assignment to discrete parents
- Most common is the linear Gaussian model, for example

$$
\begin{aligned}
& P(Cost = c \mid Harvest = h, Subsidy? = true) \\
& = \mathcal{N}(a_t h + b_t, \sigma_t)(c) \\
& = \frac{1}{\sigma_t \sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{c - (a_t h + b_t)}{\sigma_t} \right)^2 \right)
\end{aligned}
$$

- Mean *Cost* varies linearly with *Harvest*, variance is fixed
- Linear variation is unreasonable over the full range but works if the likely range of *Harvest* is narrow

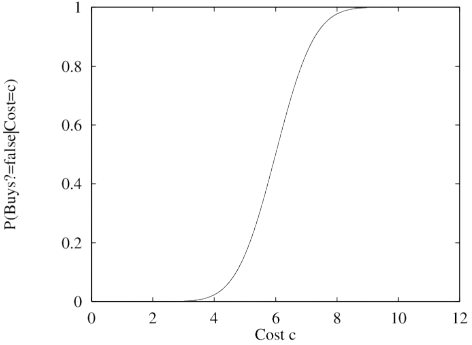
Continuous Child Variables



- All continuous network with linear Gaussian distributions $\rightarrow$ full joint distribution is a multivariate Gaussian
- Discrete + continuous linear Gaussian network is a conditional Gaussian network, that is, a multivariate Gaussian over all continuous variables for each combination of discrete values

Discrete Variable with Continuous Parents

- Probability of *Buys*? given *Cost* should be a "soft" threshold



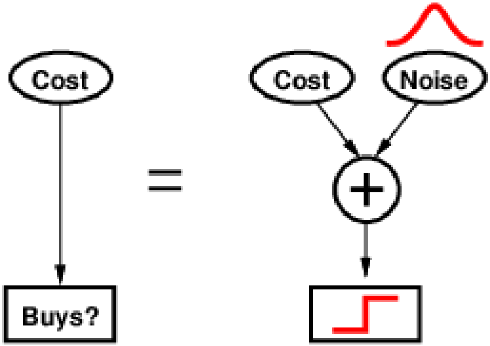
- Probit distribution uses integral of Gaussian

$$\Phi(x) = \int_{-\infty}^{x} \mathcal{N}(0,1)(x)dx$$

$$P(Buys? = true \mid Cost = c) = \Phi\left(\frac{-c+\mu}{\sigma}\right)$$

Why the Probit?

- It is sort of the right shape
- Can view as hard threshold whose location is subject to noise

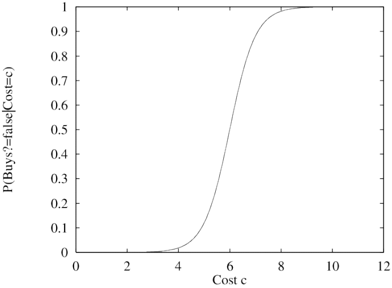


Discrete Variable Continued

- Sigmoid (or logit) distribution also used in neural networks:

$$P(Buys? = true \mid Cost = c) = \frac{1}{1 + \exp\left(-2\frac{-c+\mu}{\sigma}\right)}$$

- Sigmoid has a similar shape to probit, but much longer tails:



Inference Tasks

- Simple queries: compute posterior marginal $P(X_i \mid E = e)$
- Conjunctive queries:
  $P(X_i, X_j \mid E = e) = P(X_i \mid E = e)P(X_j \mid X_i, E = e)$
- Optimal decisions: decision networks include utility information; probabilistic inference required for
  $P(outcome \mid action, evidence)$
- Value of information: which evidence to seek next?
- Sensitivity analysis: which probability values are most critical?
- Explanation: why do I need a new starter motor?

Inference by Enumeration

- Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation
- Simple query on the burglary network:

$$
\begin{aligned}
P(B \mid j, m) &= \frac{P(B, j, m)}{P(j, m)} \\
&= \alpha P(B, j, m) \\
&= \alpha \sum_e \sum_a P(B, e, a, j, m) \\
&= \alpha \sum_e \sum_a P(B) P(e) P(a \mid B, e) P(j \mid a) P(m \mid a) \\
&= \alpha P(B) \sum_e P(e) \sum_a P(a \mid B, e) P(j \mid a) P(m \mid a)
\end{aligned}
$$

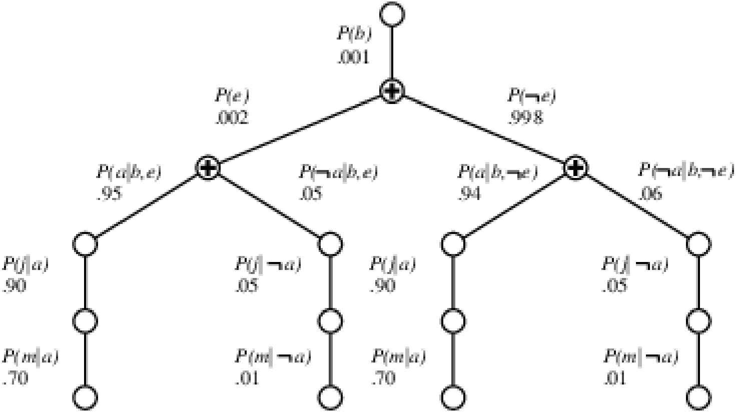
- Recursive depth-first search enumeration: $\mathcal{O}(n)$ space, $\mathcal{O}(d^n)$ time

Enumeration Algorithm

```
function ENUMERATION-ASK(X, e, bn)
    Q(X) ← a distribution over X, initially empty
    for each value x_i of X do
        extend e with value x_i for X
        Q(x_i) ← ENUMERATE-ALL(VARS[bn], e)
    return NORMALIZE(Q(X))

function ENUMERATE-ALL(vars, e)
    if EMPTY?(vars) then
        return 1.0
    Y ← FIRST(vars)
    if Y has value y in e then
        return P(y | Pa(Y) × ENUMERATE-ALL(REST(vars), e)
    else
        return ∑_y P(y | Pa(Y)) ×
ENUMERATE-ALL(REST(vars), e_y)
```

Evaluation Tree



- Enumeration is inefficient: repeated computation, for example, $P(j \mid a)P(m \mid a)$ is computed for each value of e

Inference by Variable Elimination

- Variable elimination: carry out summations right-to-left, storing intermediate results (factors) to avoid recomputation

$$
\begin{aligned}
& P(B \mid j, m) \\
& = \alpha P(B) \sum_e P(e) \sum_a P(a \mid B, e) P(j \mid a) P(m \mid a) \\
& = \alpha P(B) \sum_e P(e) \sum_a P(a \mid B, e) P(j \mid a) f_M(a) \\
& = \alpha P(B) \sum_e P(e) \sum_a P(a \mid B, e) f_J(a) f_M(a) \\
& = \alpha P(B) \sum_e P(e) \sum_a f_A(a, b, e) f_J(a) f_M(a) \\
& = \alpha P(B) \sum_e P(e) f_{\bar{A}JM}(b, e) \\
& = \alpha P(B) f_{\bar{E}\bar{A}JM}(b) \\
& = \alpha f_B(b) f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Variable Elimination: Basic Operations

- Summing out a variable from a product of factors: move any constant factors outside the summation and add up submatrices in pointwise product of remaining factors

$$
\begin{aligned}
&\sum_x f_1 \times \cdots \times f_k \\
&= f_1 \times \cdots \times f_i \sum_x f_{i+1} \times \cdots \times f_k \\
&= f_1 \times \cdots f_i \times f_{\bar{X}}
\end{aligned}
$$

  assuming $f_1, \ldots, f_i$ do not depend on X

- Pointwise product of factors f_1 and f_2:
  $f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l) =$
  $f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l)$
- Example: $f_1(a, b) \times f_2(b, c) = f(a, b, c)$

Variable Elimination Algorithm

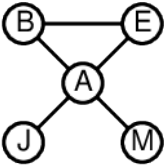
function ELIMINATION-ASK(X, e, bn)
    factors $\leftarrow$ []
    vars $\leftarrow$ REVERSE(VARS([*bn*]))
    for each *var* in *vars* **do**
        factors $\leftarrow$ [MAKE-FACTOR(*var*, *e*) | *factors*]
        if *var* is a hidden variable **then**
            factors $\leftarrow$ SUM-OUT(*vars*, *factors*)
    return NORMALIZE(POINTWISE-PRODUCT(*factors*))

Irrelevant Variables

- Consider the query $P(JohnCalls \mid Burglary = true)$
  $P(J \mid b) =$
  $\alpha P(b) \sum_e P(e) \sum_a P(a \mid b, e) P(J \mid a) \sum_m P(m \mid a)$
- The sum over m is identically 1; M is irrelevant to the query
- Theorem: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup E)$
- Here, $X = JohnCalls, E = \{Burglary\}$, and $Ancestors(\{X\} \cup E) = \{Alarm, Earthquake\}$ so $MaryCalls$ is irrelevant

Irrelevant Variables

- Definition: moral graph of Bayes net – marry all parents and drop arrows
- Definition: A is *m-separated* from B by C iff separated by C in the moral graph
- Theorem: Y is irrelevant if m-separated from X by E
- Example: For $P(JohnCalls \mid Alarm = true)$, both *Burglary* and *Earthquake* are irrelevant



Complexity of Exact Inference

- Singly connected networks (or polytrees):
  - any two nodes are connected by at most one (undirected) path
  - time and space cost of variable elimination are $\mathcal{O}(d^k n)$
- Multiply connected networks:
  - can reduce to 3SAT to exact inference $\Rightarrow$ NP-hard
  - equivalent to counting 3SAT models $\Rightarrow$ P-complete

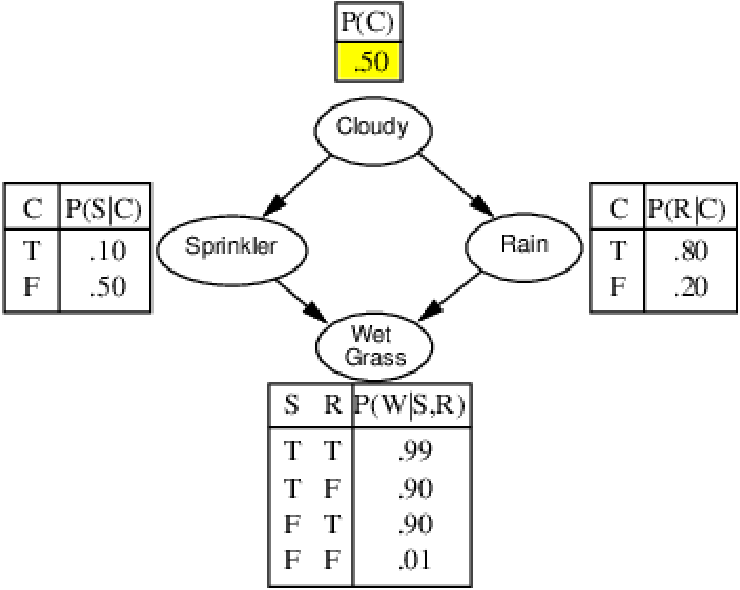
Inference by Stochastic Simulation

- Basic idea:
  1. Draw N samples from a sampling distribution S
  2. Compute an approximate posterior probability $\hat{P}$
  3. Show this converges to the true probability P
- Outline
  - Sampling from an empty network
  - Rejection sampling: reject samples that disagree with the evidence
  - Likelihood weighting (LW): use evidence to weight samples
  - Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an Empty Network

function PRIOR-SAMPLE(*bn*)
    $x \leftarrow$ an event with n elements
    for $i = 1$ to n **do**
        $x_i \leftarrow$ a random sample from $P(X_i \mid parents(X_i))$
            given the values of $Parents(X_i)$ in x

Example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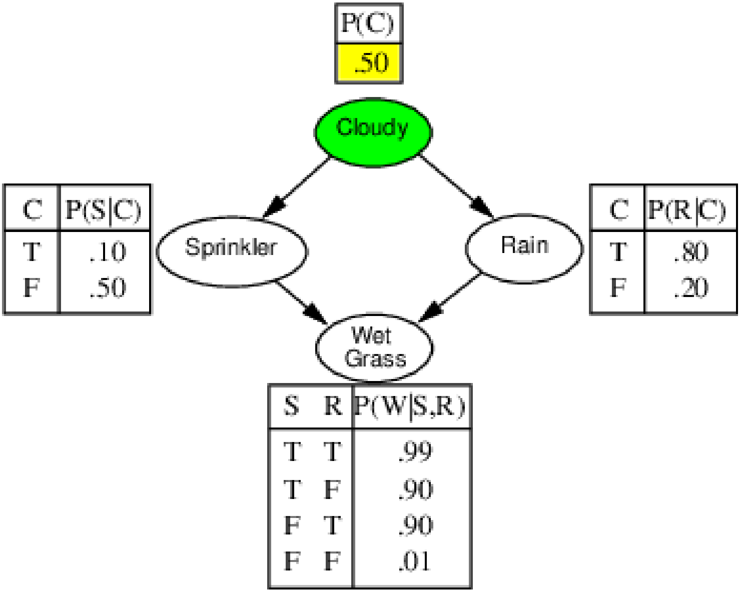

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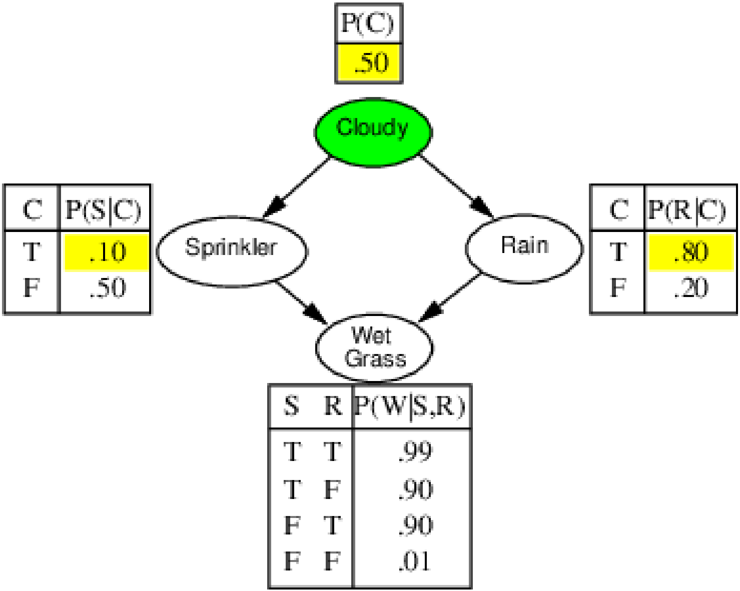

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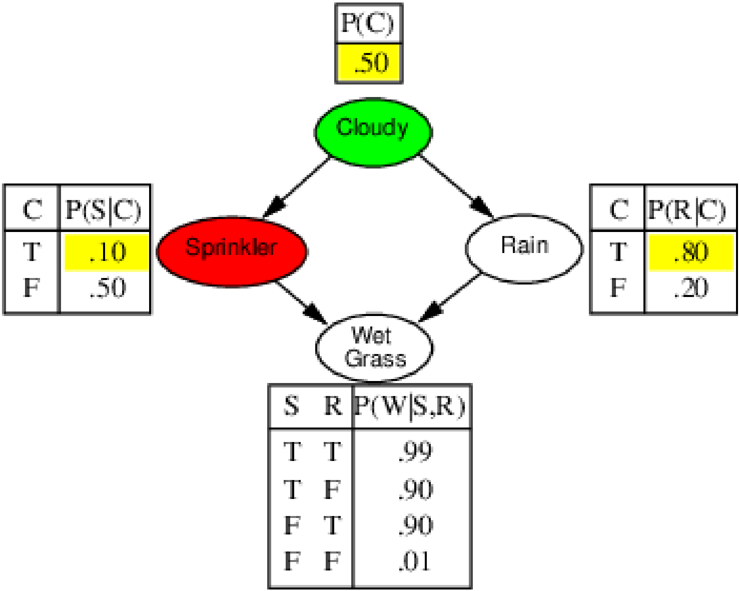
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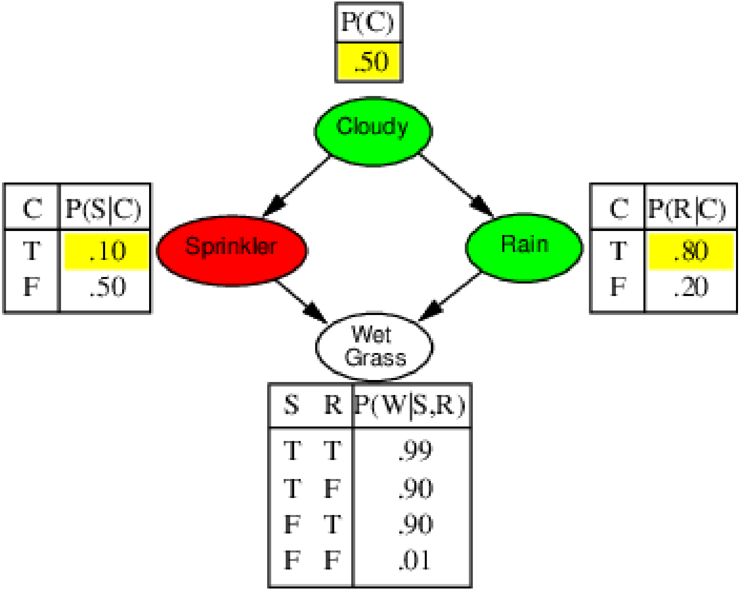
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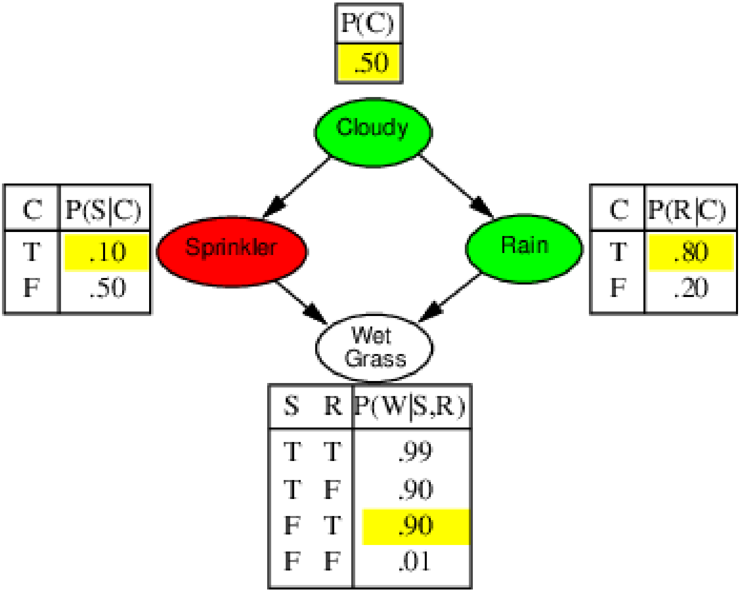
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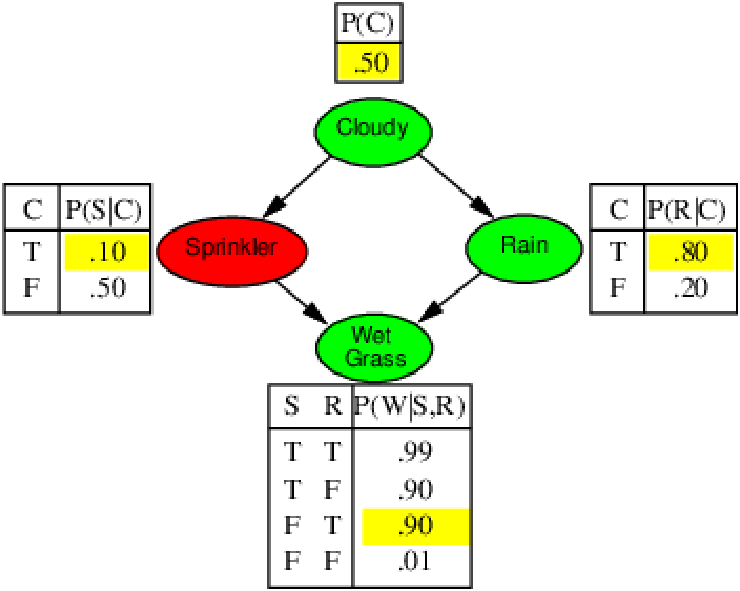
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Sampling from an Empty Network Continued

- Probability the PRIOR-SAMPLE generates a particular event

$$S_{PS}(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(X_i)) = P(x_1, \ldots, x_n)$$

  that is, the true prior probability

- Let $N_{PS}(x_i, \ldots, x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$, then we have

$$\begin{aligned}\lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} \frac{N_{PS}(x_1, \ldots, x_n)}{N} \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1, \ldots, x_n)\end{aligned}$$

- Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1, \ldots, x_n)$

Rejection Sampling

- $\hat{P}(X \mid e)$ estimated from samples agreeing with e
- Example: estimate $P(Rain \mid Sprinkler = true)$ using 100 samples: 27 samples have $Sprinkler = true$ and of these 8 have $Rain = true$ and 19 have $Rain = false$.
- $\hat{P}(Rain \mid Sprinkler = true) = \textsc{Normalize}(\langle 8, 19 \rangle) = \langle 0.296, 0.704 \rangle$

Analysis of Rejection Sampling

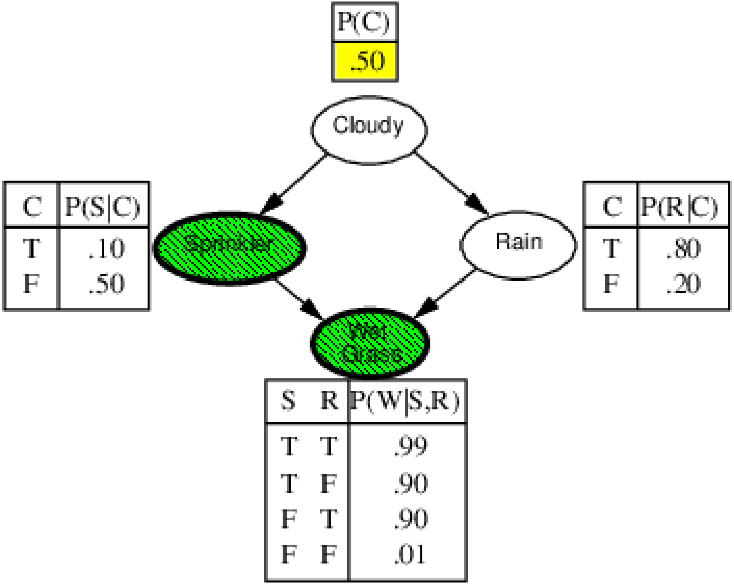
$$\begin{aligned}\hat{P}(X \mid e) &= \alpha N_{PS}(X, e)\\ &= \frac{N_{PS}(X, e)}{N_{PS}(e)}\\ &\approx \frac{P(X, e)}{P(e)}\\ &= P(X \mid e)\end{aligned}$$

- Hence rejection sampling returns consistent posterior estimates
- Problem: hopelessly expensive if $P(e)$ is small
- $P(e)$ drops off exponentially with number of evidence variables

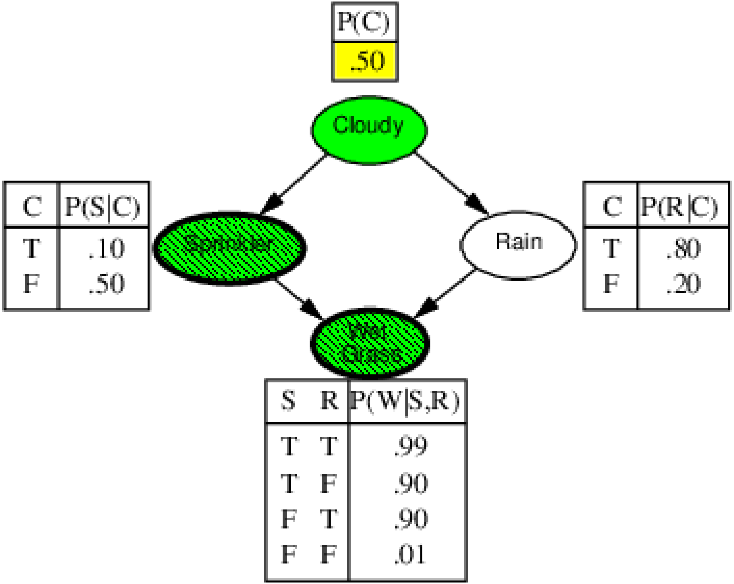
Likelihood Weighting

- Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

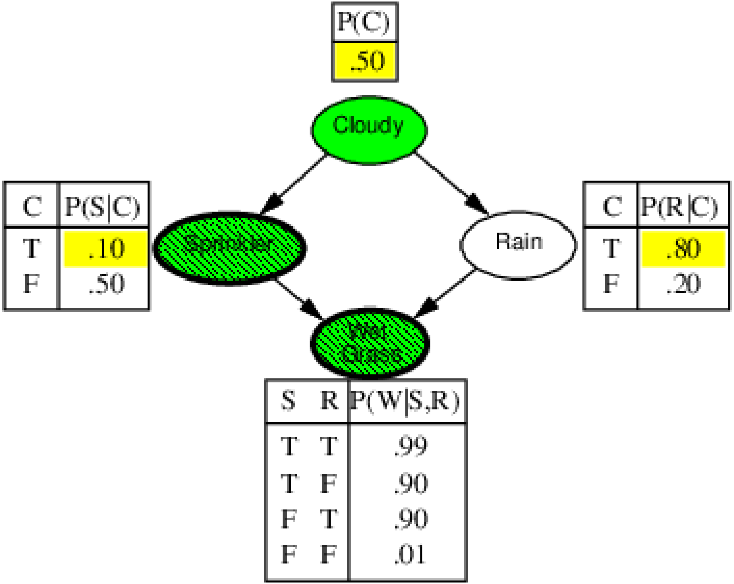
Likelihood Weighting Example



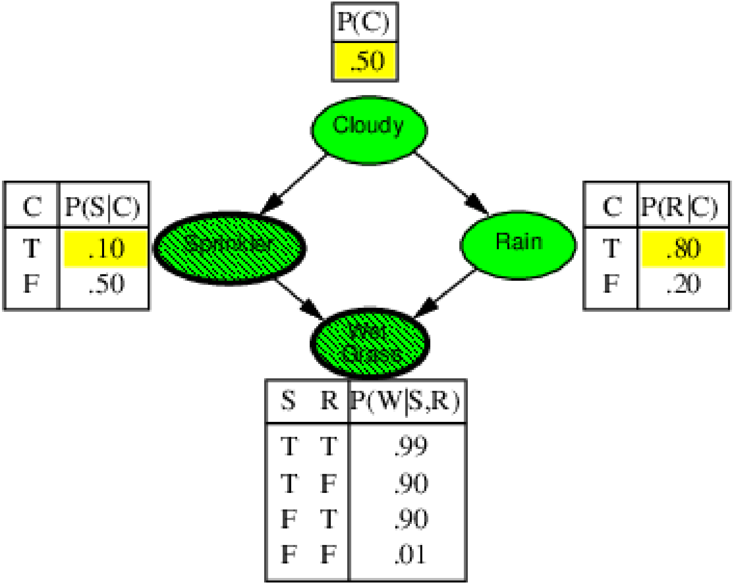
Likelihood Weighting Example



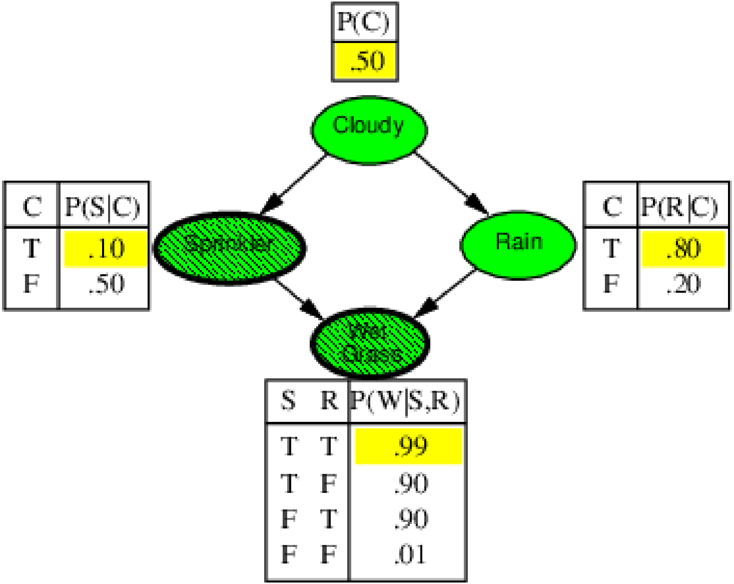
Likelihood Weighting Example



Likelihood Weighting Example



Likelihood Weighting Example



Likelihood Weighting Analysis

- Sampling probability for WEIGHTED-SAMPLE is
  $S_{WS}(z,e) = \prod_{i=1} lP(z_i \mid parents(Z_i))$

- Note: pays attention to evidence in *ancestors* only –
  somewhere "in between" prior and posterior distribution

- Weight for a given sample z, e is
  $w(z,e) = \prod_{i=1}^{m} P(e_i \mid parents(E_i))$

- Weighted sampling probability is

$$
\begin{aligned}
&S_{WS}(z,e)w(z,e) \\
&\qquad = \prod_{i=1}^{l} P(z_i \mid parents(Z_i)) \prod_{i=1} mP(e_i \mid parents(E_i)) \\
&\qquad = P(z,e) \text{ (by standard global semantics of network)}
\end{aligned}
$$

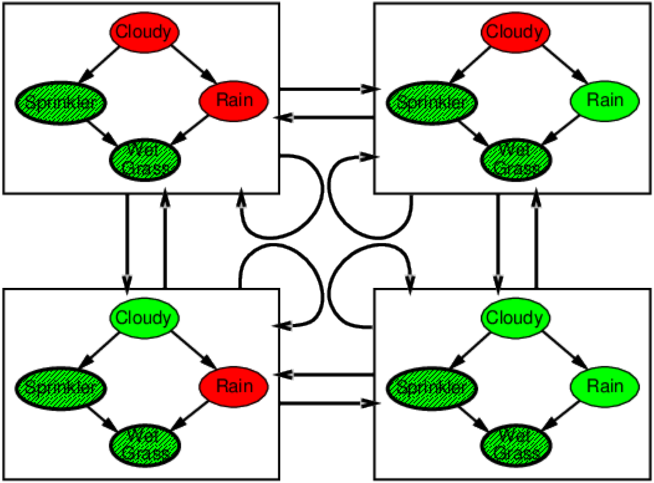
- Hence likelihood weighting returns consistent estimates but
  performance still degrades with many evidence variables
  because few samples have nearly all the total weight

Approximate Inference Using MCMC

- "State" of network is current assignment to all variables
- Generate next state by sampling one variable given Markov blanket

The Markov Chain

- With $Sprinkler = true$, $WetGrass = true$, there are four states:



- Wander about for a while, average what yo usee

MCMC Example Continued

- Estimate $P(\mathit{Rain} \mid \mathit{Sprinkler} = \mathit{true}, \mathit{WetGrass} = \mathit{true})$
- Sample *Cloudy* or *Rain* given its Markov blanket, repeat; count the number of times *Rain* is true and false in the samples.
- For example, visit 100 states: 31 have $\mathit{Rain} = \mathit{true}$, 69 have $\mathit{Rain} = \mathit{false}$
  $\hat{P}(\mathit{Rain} \mid \mathit{Sprinkler} = \mathit{true}, \mathit{WetGrass} = \mathit{true}) =$
  $\textsc{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$
- Theorem: chain approaches stationary distribution: long-run fraction of time spent in each state is exactly proportional to its posterior probability.

Markov Blanket Sampling

- Markov blanket of *Cloudy* is *Sprinkler* and *Rain*
- Markov blanket of *Rain* is *Cloudy*, *Sprinkler*, and *WetGrass*
- Probability given the Markov blanket is calculated as follows:

$$P(x_i' \mid mb(X_i)) = P(x_i' \mid parents(X_i)) \prod_{Z_j \in Children(X_i)} P(z_j \mid parents(Z_j))$$

- Easily implemented in message-passing parallel systems
- Main computational problems:
  1. Difficult to tell if convergence has been achieved
  2. Can be wasteful if Markov blanket is large: $P(X_i \mid mb(X_i))$ will not change much (law of large numbers)

Summary

- Bayes nets provide a natural representation for (causally induced) conditional independence
- Topology + CPTs = compact representation for joint distribution
- Generally easy for (non)experts to construct
- Canonical distributions (e.g. noisy-OR) = compact representation of CPTs
- Continuous variables $\rightarrow$ parameterized distributions
- Exact inference by variable elimination: NP-hard in general
- Approximate inference methods: likelihood weighting and Markov chain Monte-Carlo sampling