Rational Decisions

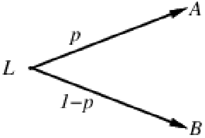
CSC 548, Artificial Intelligence II

Preferences

- An agent chooses among *prizes* (A, B, etc.) and *lotteries* (situations with uncertain prizes).
- Preference Notation:

$A \succ B$	A preferred to B
$A \backsim B$	indifference between A and B
$A \succsim B$	B not preferred to A

- Lottery notation: $L = [p, A; (1 - p), B]$

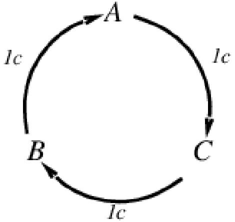


Rational Preferences

- Idea: preferences of a rational agent must obey constraints
- Rational preferences $\Rightarrow$ behavior describable as maximization of expected utility.
- Constraints:
  - Orderability:
    $(A \succ B) \vee (B \succ A) \vee (A \backsim B)$
  - Transitivity:
    $(A \succ B) \wedge (B \succ C) \rightarrow (A \succ C)$
  - Continuity:
    $A \succ B \succ C \rightarrow \exists p[p, A; 1-p, C] \backsim B$
  - Substitutability:
    $A \backsim B \rightarrow [p, A; 1-p, C] \backsim [p, B; 1-p, C]$
  - Monotonicity:
    $A \succ B \rightarrow (p \geq q \leftrightarrow [p, A; 1-p, B] \succsim [q, A; 1-q, B])$

Rational Preferences

- Violating the constraints leads to self-evident irrationality
- For example: an agent with intransitive preferences can be induced to give away all its money
  - If $B \succ C$, then an agent who has C would pay (say) 1 cent to get B
  - If $A \succ B$, then an agent who has B would pay (say) 1 cent to get A
  - If $C \succ A$, then an agent who has A would pay (say) 1 cent to get C



Maximizing Expected Utility

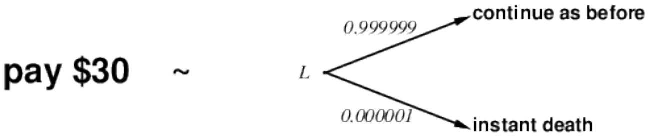
- Theorem (Ramsey, 1931; von Neumann and Morgenstern 1944): Given preferences satisfying the constraints there exists a real-valued function U such that

$$U(A) \geq U(B) \leftrightarrow A \succsim B$$
$$U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)$$

- Maximum Expected Utility (MEU) principle: choose the action that maximizes expected utility
- Note: an agent can be entirely rational (consistent with MEU) without ever representing of manipulating utilities and probabilities

Utilities

- Utilities map states to real numbers
- Standard approach to assessment of human utilities:
  - compare a given state A to a *standard lottery* L_p that has "best possible prize" $u_\top$ with probability p and "worst possible catastrophe" $u_\perp$ with probability $(1 - p)$
  - adjust lottery probability p until $A \backsim L_p$



Utility Scales

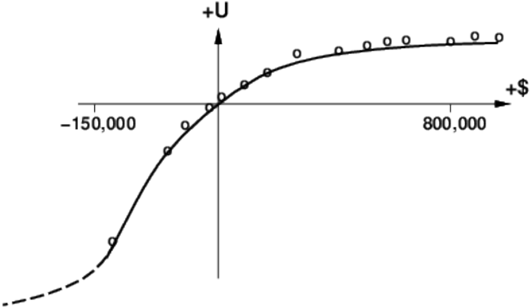
- Normalized utilities: $u_\top = 1.0, u_\bot = 0.0$
- Micromorts: one-millionth chance of death
  useful for Russian roulette, paying to reduce risks, etc.
- QALYs: quality-adjusted life years
  useful for medical decisions involving substantial risk
- Note: behavior is invariant with respect to $+ve$ linear transformation

$$U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0$$

- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, that is, total order on prizes

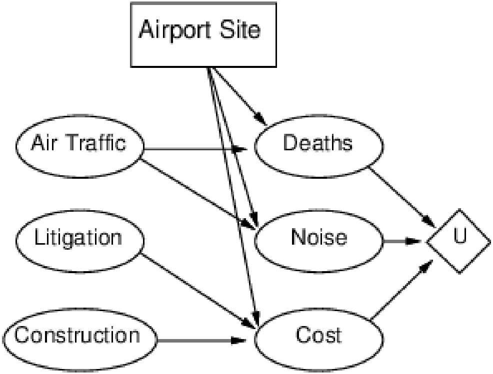
Money

- Money does **not** behave as a utility function
- Given a lottery L with expected monetary value $EMV(L)$, usually $U(L) < U(EMV(L))$, that is, people are risk-averse
- Utility curve: for what probability p am I indifferent between prize x and a lottery $[p, \$M; (1-p), \$0]$ for large M?
- Typical empirical data, extrapolated with risk-prone behavior:



Decision Networks

- Add *action nodes* and *utility nodes* to belief networks to enable rational decision making



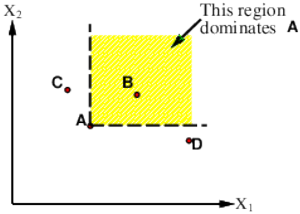
- Algorithm:
  - For each value of action node, compute expected value of utility node given action, evidence

Multiattribute Utility

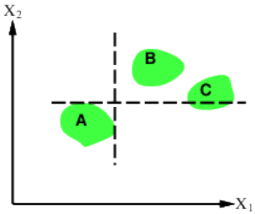
- How can we handle utility functions of many variable $X_1 \ldots X_n$?
- For example, what is $U(\mathit{Deaths}, \mathit{Noise}, \mathit{Cost})$
- How can complex utility functions be assessed from preference behavior?
- Idea 1: identify conditions under which decisions can be made without complete identification of $U(x_1, \ldots, x_n)$
- Idea 2: identify various types of independence in preferences and derive consequent canonical forms for $U(x_1, \ldots, x_n)$

Strict Dominance

- Typically define attributes such that U is monotonic in each
- Strict dominance: choice B strictly dominates choice A iff $\forall i\ X_i(B) \geq X_i(A)$ (and hence $U(B) \geq U(A)$)



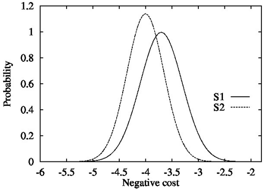
Deterministic attributes

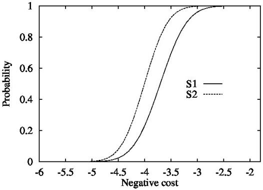


Uncertain attributes

- Strict dominance seldom holds in practice

Stochastic Dominance





- Distribution p_1 stochastically dominates distribution p_2 iff

$$\forall\, t \int_{-\infty}^{t} p(x)dx \ \leq \int_{-\infty}^{t} p_2(x)d(x)$$

- If U is monotonic in x, then A_1 with outcome distribution p_1 stochastically dominates A_2 with outcome distribution p_2:

$$\int_{-\infty}^{\infty} p_1(x)U(x)d(x) \geq \int_{-\infty}^{\infty} p_2(x)U(x)dx$$

- Multiattribute case: stochastic dominance on all attributes $\Rightarrow$ optimal

Stochastic Dominance

- Stochastic dominance can often be determined without exact distributions using qualitative reasoning
- For example, construction cost increases with distance from city: S_1 is closer to the city than $S_2 \rightarrow S_1$ stochastically dominates S_2 on cost
- For example, injury increases with collision speed
- Can annotate belief networks with stochastic dominance information: $X \stackrel{+}{\rightarrow} Y$ (X positively influences Y) means that for every value z of Y's other parents Z
$\forall\, x_1, x_2 \geq x_2 \rightarrow P(Y \mid x_1, z)$ stochastically dominates $P(Y \mid x_2, z)$

Preference Structure: Deterministic

- X_1 and X_2 preferentially independent (P.I.) of X_3 iff preference between $\langle x_1, x_2, x_3 \rangle$ and $\langle x_1', x_2', x_3' \rangle$ does not depend on x_3
- For example, $\langle Noise, Cost, Safety \rangle$:
  ⟨ 20,000 suffer, \$4.6 billion, 0.06 deaths/mpm ⟩ versus
  ⟨ 70,000 suffer, \$4.2 billion, 0.06 deaths/mpm ⟩
- Theorem (Leontief, 1947): if every pair of attributes is P.I. of its complement, then every subset of attributes is P.I. of its complement: mutual P.I.
- Theorem (Debreu, 1960): mutual P.I. $\rightarrow \exists$ additive value function:

  $$V(S) = \sum_i V_i(X_i(S))$$

  Hence assess n single-attribute functions; often a good approximation

Preference Structure: Stochastic

- Need to consider preferences over lotteries: X is *utility-independent* of Y iff preferences over lotteries in X do not depend on y
- Mutual P.I.: each subset is U.I. of its complement $\rightarrow \exists$ multiplicative utility function:

$$\begin{aligned} U &= k_1 U_1 + k_2 U_2 + k_3 U_3 \\ &\quad + k_1 k_2 U_1 U_2 + k_2 k_3 U_2 U_3 + k_3 k_1 U_3 U_1 \\ &\quad + k_1 k_2 k_3 U_1 U_2 U_3 \end{aligned}$$

- Routine procedures and software packages for generating preference tests to identify various canonical families of utility functions

Value of Information

- Idea: compute value of acquiring each possible piece of evidence; can be done directly from the decision network
- Example: buying oil drilling rights
  - two blocks A and B, exactly one has oil, worth k
  - prior probabilities 0.5 each, mutually exclusive
  - current price of each block $k/2$
  - "consultant" offers accurate survey of A – fair price?
- Solution: compute the expected value of information – expected value of the best action given the information minus expected value of best action without information
- Survey may say "oil in A" or "no oil in A"
  $= [0.5\times$ value of "buy A" given "oil in $A +$
  $0.5\times$ value of"buy B" given "no oil in A"] - 0
  $= (0.5 \times k/2) + (0.5 \times k/2) - 0 = k/2$

General Formula

- Current evidence E, current best action α, possible action outcomes S_i, potential new evidence E_j

$$EU(\alpha \mid E) = \max_a \sum_i U(S_i) P(S_i \mid E, a)$$

- Suppose we knew $E_j = e_{jk}$, then we would choose $\alpha_{e_{jk}}$ s.t.

$$EU(\alpha_{e_{jk}} \mid E, E_j = e_{jk}) = \max_a \sum_i U(S_i) P(S_i \mid E, a, E_j = e_{jk})$$

- E_j is a random variable whose value is *currently* unknown $\Rightarrow$ must compute expected gain over all possible values:

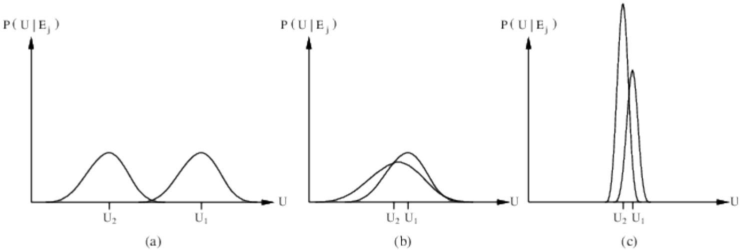
$$VPI_E(E_j) = \left(\sum_k P(E_j = e_{jk} \mid E) EU(\alpha_{e_{jk}} \mid E, E_j = e_{jk})\right) - EU(\alpha \mid E)$$

(VPI = value of perfect information)

Properties of VPI

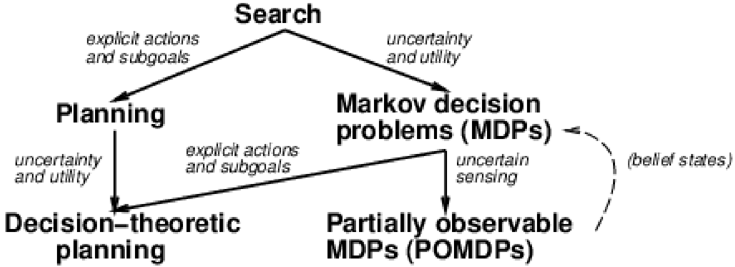
- Nonnegative (in expectation)
  $\forall j, E\ VPI_E(E_j) \geq 0$
- Nonadditive (consider obtaining E_j twice)
  $VPI_E(E_j, E_k) \neq VPI_E(E_j) + VPI_E(E_k)$
- Order-independent
  $VPI_E(E_j, E_k) = VPI_E(E_j) + VPI_{E,E_j}(E_k) =$
  $VPI_E(E_k) + VPI_{E,E_k}(E_j)$
- Note: when more than one piece of evidence can be gathered, maximizing VPI for each to select one is not always optimal $\Rightarrow$ evidence-gathering becomes a sequential decision problem

Qualitative Behaviors

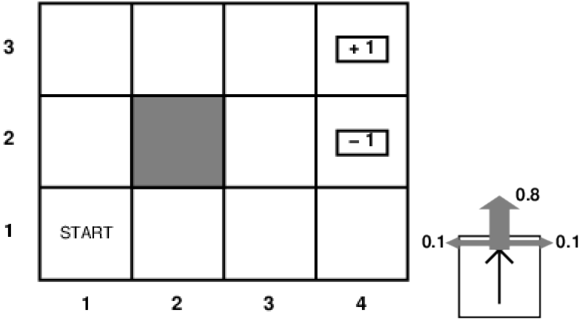


- a: choice is obvious, information worth little
- b: choice is nonobvious, information worth a lot
- c: choice is nonobvious, information worth little

Sequential Decision Problems



Example Markov Decision Process (MDP)

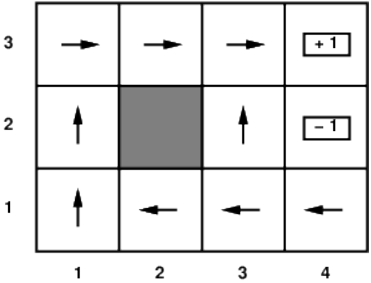


- States $s \in S$, actions $a \in A$
- Model: $T(s, a, s') \equiv P(s' \mid s, a) =$ probability that a in s leads to s'
- Reward function:

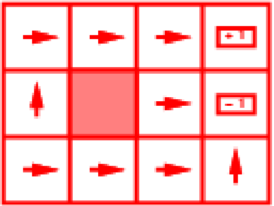
$$R(a) = \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

Solving Markov Decision Processes

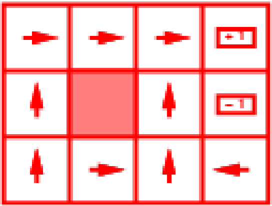
- In search problems, aim is to find an optimal sequence
- In MDPs, aim is to find optimal policy $\pi(s)$: best action for every possible state s (because we cannot predict where one will end up)
- The optimal policy maximizes (say) the *expected sum of rewards*
- Optimal policy when state penalty $R(s)$ is -0.04:



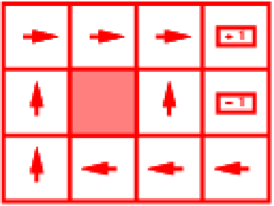
Risk and Reward



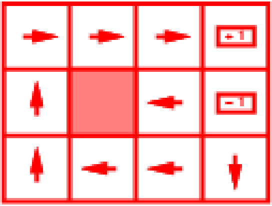
r = [$-\infty$: −1.6284]



r = [−0.4278 : −0.0850]



r = [−0.0480 : −0.0274]



r = [−0.0218 : 0.0000]

Utility of State Sequences

- Need to understand preferences between *sequences* of states
- Typically consider stationary preferences on reward sequences:

  $$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots] \leftrightarrow [r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- Theorem: there are only two ways to combine rewards over time:

  1. *Additive* utility function:

     $$U([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \ldots$$

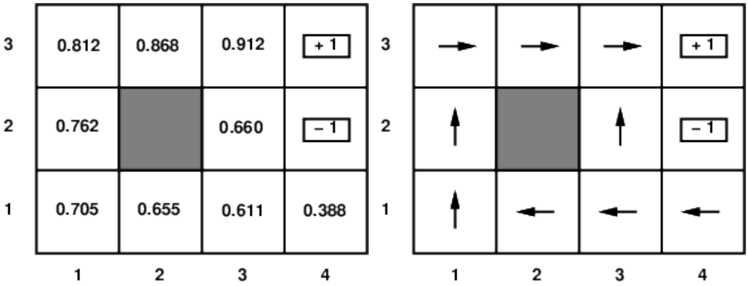
  2. *Discounted* utility function:

     $$U([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

     where γ is the discount factor.

Utility of States

- Utility of a state (a.k.a. its value) is defined to be $U(s) =$ expected (discounted) sum of rewards (until termination) assuming optimal actions
- Given the utilities of the states, choosing the best action is just MEU: maximize the expected utility of the immediate successors



Utilities

- Problem: infinite lifetimes $\Rightarrow$ additive utilities are infinite

1. Finite Horizon: termination at a *fixed time* T $\Rightarrow$ nonstationary policy: $\pi(s)$ depends on time left
2. Absorbing state(s): with probability 1, agent eventually "dies" for any *pi* $\Rightarrow$ expected utility of every state is finite
3. Discounting: assuming $\gamma < 1, R(s) \leq R_{\max}$,

   $$U([s_0, \ldots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq R_{\max}/(1-\gamma)$$

   smaller $\gamma \Rightarrow$ shorter horizon
4. Maximize system gain $=$ average reward per time step:
   Theorem: optimal policy has constant gain after intial transient

Dynamic Programming: the Bellman Equation

- Definition of utility of states leads to a simple relationship among utilities of neighboring states: expected sum of rewards $=$ current reward $+ \gamma \times$ expected sum of rewards after taking best action
- Bellman equation (1957):

$$U(s) = R(s) + \gamma \max_a \sum_{s'} U(s')T(s, a, s')$$

- Example:

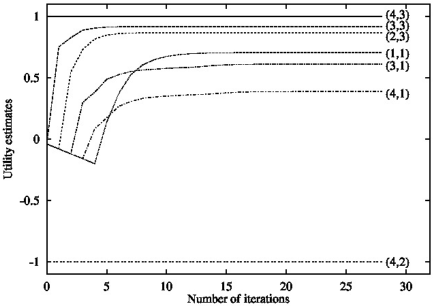
$$\begin{aligned} U(1,1) = -0.04 + \gamma \max(& \\ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), \\ & 0.9U(1,1) + 0.1U(1,2), \\ & 0.9U(1,1) + 0.1U(2,1), \\ & 0.8U(2,1) + 0.1U(1,2), 0.1U(1,1) \end{aligned}$$

Value Iteration Algorithm

- Idea: start with arbitrary utility values
  Update to make them *locally consistent* with Bellman equation
  Everywhere locally consistent $\Rightarrow$ global optimality

- Repeat for every s simultaneously until "no change"

$$U(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} U(s')T(s,a,s') \quad \forall\, s$$



Convergence

- Define the max-norm $\|U\| = \max_s |U(s)|$, so $\|U - V\| =$ maximum difference between U and V
- Let U^t and U^{t+1} be successive approximations to the true utility
- Theorem: for any two approximations U^t and V^t

$$\|U^{t+1} - V^{t+1}\| \leq \|U^t - V^t\|$$

  That is, any distinct approximations must get closer to each other so, inparticular, any approximation must get closer to the true U and value iteration converges to a unique, stable optimal solution
- Theorem: if $\|U^{t+1} - U^t\| < \epsilon$, then $\|U^{t+1} - U\| < \frac{2\epsilon\gamma}{1-\gamma}$ That is, once the change in U^t becomes small, we are almost done
- MEU policy using U^t may be optimal long before convergence of values

Policy Iteration

- Howard, 1960: search for optimal policy and utility values simultaneously
- To compute utilities given a fixed π (value determination):

$$U(s) = R(s) + \gamma \sum_{s'} U(s') T(s, \pi(s), s') \quad \forall s$$

  That is, n simultaneous linear equations in n unknowns, solve in $\mathcal{O}(n^3)$

Modified Policy Iteration

- Policy iteration often converges in few iterations, but each is expensive
- Idea: use a few steps of value iteration (but with π fixed) starting from the value function produced the last time to produce an approximate value determination step
- Often converges much faster than pure value iteration or policy iteration
- Leads to much more general algorithms where Bellman value updates and Howard policy updates can be performed locally in any order
- Reinforcement learning algorithms operate by performing such updates based on the observed transitions made in an initially unknown environment

Partial Observability

- A Partially Observable Markov Decision Process (POMP) has an *observation model* $O(s, e)$ defining the probability that the agent obtains evidence e when in state s
- Agent does not know which state it is in $\Rightarrow$ makes no sense to talk about policy π
- Theorem (Astrom 1965): the optimal policy in a POMPD is a function $\pi(b)$ where b is the *belief state* (probability distribution over states)
- Can convert a POMPD into an MDP in belief-state space, where $T(b, a, b')$ is the probability that the new belief state is b' given that the current belief state is b and the agent does a

Partial Observability

- Solutions automatically include information-gathering behavior
- If there are n states, b is an n-dimensional real-valued vector
  $\Rightarrow$ solving POMPDs is very (actually, PSPACE) hard
- The real world is a POMDP (with initially unknown T and O