Propositional Logic

CPSC 447 - Artificial Intelligence I

Propositions

- A *proposition* is a declarative sentence that is either true or false

Propositional Variables

- A *propositional variable (p, q, r, s, ...)* is a mathematical variable representing a proposition
- The value of a propositional variable is true, denoted by **T**, or false, denoted by **F**

Compound Propositions

- A *compound proposition* is a proposition constructed by combining propositions with logical operators
- Logical operators:
  - $\neg$: Negation
  - $\vee$: Disjunction
  - $\wedge$: Conjunction
  - $\oplus$: Exclusive Or
  - $\rightarrow$: Conditional
  - $\leftrightarrow$: Biconditional

Truth Tables

- A *truth table* is used to summarize some or all of the possible values of one or more propositions in conjunction with any number of logical operations on those propositions.

Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has the following truth table:

p	$\neg p$
T	F
F	T

Conjunction

- The *conjunction* of a propositions p and q is denoted by $p \wedge q$ and has the following truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

- The *exclusive or* of propositions p and q is denoted by $p \oplus q$ and has the following truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- The *conditional statement* or *implication* of propositions p and q is denoted by $p \to q$ and has the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- $p \to q$ is read "If p then q"
- In $p \to q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*)

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements
  - $q \to p$ is the *converse* of $p \to q$
  - $\neg q \to \neg p$ is the *contrapositive* of $p \to q$
  - $\neg p \to \neg q$ is the *inverse* of $p \to q$

Biconditional

- The *biconditional* of propositions p and q is denoted by $p \leftrightarrow q$ and has the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- $p \leftrightarrow q$ is read "p if and only if q"

Truth Tables for Compound Propositions

- Truth table construction:
  - We need a row for every possible combination of truth values for the atomic propositions
  - We need a column for the compound proposition
  - We need a column for each subexpression (including the atomic propositions)

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value
- Example: the conditional is equivalent to the contrapositive

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Tautologies, Contradictions, and Contingencies

- A *tautology* is a proposition that is always true
  - Example: $p \lor \neg p$
- A *contradiction* is a proposition that is always false
  - Example: $p \land \neg p$
- A *contingency* is a proposition that is neither a tautology nor a contradiction

Logic Equivalence

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology
- This is denoted as $p \equiv q$
- Logical equivalence can be shown with a truth table; the compound propositions p and q are equivalent if and only if the columns in the truth table agree

De Morgan's Laws

- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $\neg(p \lor q) \equiv \neg p \land \neg q$
- Truth table for second law:

p	q	$\neg p$	$\neg q$	$p \lor q$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

Identity Laws:	$p \wedge T \equiv p, \quad p \vee F \equiv p$
Domination Laws:	$p \vee T \equiv T, \quad p \wedge F \equiv F$
Idempotent Laws:	$p \vee p \equiv p, \quad p \wedge p \equiv p$
Double Negation Law:	$\neg(\neg p) \equiv p$
Negation Laws:	$p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$
Commutative Laws:	$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$
Associative Laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws:	$(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
	$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption Laws:	$p \vee (p \wedge q) \equiv p$
	$p \wedge (p \vee q) \equiv p$

Logical Equivalences Involving Conditional Statements

- $p \to q \equiv \neg p \vee q$
- $p \to q \equiv \neg q \to \neg p$
- $p \vee q \equiv \neg p \to q$
- $p \wedge q \equiv \neg(p \to \neg q)$
- $\neg(p \to q) \equiv p \wedge \neg q$
- $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
- $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
- $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
- $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Equivalence Proofs

- A compound proposition can be replaced by a logically equivalent compound proposition without changing its truth value
- We can show that two propositions are logically equivalent by developing a series of logically equivalent statements
- To prove that $A \equiv B$, we can develop a series of equivalences beginning with A and ending with B:

$$\begin{aligned} A &\equiv A_1 \\ &\equiv A_2 \\ &\vdots \\ &\equiv B \end{aligned}$$

Equivalence Proof Example

Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by De Morgan's law} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{by De Morgan's law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{by the negation law} \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law}
\end{aligned}
$$

Equivalence Proof Example

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by} p \rightarrow q \equiv \neg p \vee q \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by De Morgan's law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and} \\
&\vdots && \text{commutative laws} \\
&\equiv T \vee T && \text{by the negation law} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$