The Role of Semantic Support and Equation Format in Algebra Problem Solving

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A simple word problem that provides important semantic support can be easier for subjects to solve than a corresponding equation. This finding has been interpreted as suggesting that it might be better to teach such word problems before teaching equations, the reverse of what is typically done. However, performance on such word problems needs to be compared to equations in different formats. In a first experiment, we replicated the finding of good performance on a word problem relative to poor performance on a start-unknown equation presented alone. We extended this result by finding that, contrary to our expectation, performance on a result-unknown equation was similarly poor when it was presented alone, although any word problem subject who generated a result-unknown equation solved it correctly. Also, many errors committed by subjects trying to solve either equation alone were due to attempting to work from left to right when that would lead to violating the order of operations. In a second experiment we found that subjects were much better at solving a result-unknown equation when it was congruent with the order of operations. These findings suggest that the benefit of teaching some kinds of simple word problems before equations may hold only for certain equation formats, and there may also be a benefit of teaching some formats of equations before others.

Koedinger, Alibali, and Nathan (2008) showed that, contrary to what is often believed by both mathematics students and teachers, a word problem can sometimes be easier for students to solve than a corresponding equation because the word problem can provide important semantic support. They interpreted this finding as suggesting that it might be better to teach such word problems before teaching equations, the reverse of what is typically done. However, in Koedinger et al. (2008), the corresponding equation was in a start-unknown format (e.g., $(X - 64) \div 3 = 20.50$), a format that is usually harder than a result-unknown format (e.g., $64 + 3(20.50) = X$). Therefore, we thought that performance on such word problems ought to be compared to an equation in the easier result-unknown format.

The word problem on which Koedinger, et al. (2008) found superior performance compared to a corresponding equation was:

> *Mom won some money in a lottery. She kept $64 for herself and gave each of her three sons an equal portion of the rest of it. If each son got $20.50, how much did Mom win?*

They compared performance on that problem to performance on this start-unknown equation:

$$(X - 64) \div 3 = 20.50$$

They reported that 83% of their subjects solved the word problem compared to only 23% for the start-unknown equation, and they attributed the difference to the semantic support provided by the word problem. We were skeptical of that result for two reasons. First, from conversations with math educators we learned that, from their experience teaching word problems, they doubted that their students would perform as well on the word problem as Koedinger, et al.'s subjects (Glenna Gebhard and Amadou Guisse, personal communication). Second, Koedinger et al. regarded an equation such as $(X - 64) \div 3 = 20.50$, which they called the "(x - c) / n = y" form to be less familiar than one in what they called the "mx + b = y" form, such as $7X + .12 = 2.57$. We reasoned that if $(X - 64) \div 3 = 20.50$ was less familiar than $7X + .12 = 2.57$, both of which were start-unknown forms, then it ought to be even more unfamiliar than the usually easier result-unknown form, such as $64 + 3(20.50) = X$. Thus, we hypothesized that performance on the word problem might not be superior to performance on a corresponding equation if the equation were in the form: $64 + 3(20.50) = X$.

Another idea expressed by Koedinger, et al. (2008) led us to consider another potentially informative way to modify the procedure they had used. In their experiment, they had instructed subjects who were given the word problem to solve it in whatever way they could. They found that many subjects chose to use informal solution methods, rather than to try to set up an equation. Therefore, in order to effectively compare performance on the word problem alone to performance on the corresponding equation, they had to have another condition in which they *gave* subjects the equation to solve. But, they pointed out that the question about how the problem representation affected performance is a question about the representations people use, as opposed to the representations they are *given*. Accordingly, we thought it would be useful to try to encourage subjects to generate equations in order to be able to compare their performance on an equation that they generated themselves to performance on an equation that they were given. Our intuition was that, first, if people were presented with the word problem and asked to generate an equation, not everyone would be able to do so. Secondly, however, we believed that among those who were able to generate an equation, some would generate the easier result-unknown form, $64 + 3(20.50) = X$. We believed that such a form would be, like the “mx +b = y” form, more familiar to them. We hypothesized that those who did not generate an equation and those who generated the start-unknown form would be less likely to solve the problem. However, we also hypothesized that those who generated the result-unknown form, although they might be in the minority, would be more likely to solve the problem. Accordingly, we also hypothesized that if people were given an equation only, they would be more likely to solve the result-unknown form than the start-unknown form.

To test these hypotheses, we designed an experiment in which we randomly assigned subjects to receive either the word problem alone (with instructions to try to generate an equation, if possible), the start-unknown equation alone, or the result-unknown equation alone.

Experiment 1

Method

Participants. The participants were 58 college students at a 4 year undergraduate teaching university. They were recruited from a variety of five different summer session classes. Two of the classes were in the Mathematics department. They were an introductory math course and a college algebra course. Three of the classes were in the Psychology department. They were an abnormal psychology, a personality, and a statistics course.

Materials and procedure. In the word problem condition (N = 20), we gave the subjects the simple word problem shown in the introduction. We instructed the subjects to try to generate an equation first, and then, regardless of whether or not they were able to generate an equation, to try to solve the problem. In the start-unknown condition (N = 19), we gave the subjects the equation $(X - 64) \div 3 = 20.50$ and instructed them to try to solve the equation, showing all their work. In the result-unknown condition (N = 19) we gave them $64 + 3(20.50) = X$ with the same instructions. The subjects were instructed to work independently and not look at anyone else's work. They were not timed.

Results

Scoring. In all conditions we scored the participant for correctly solving the problem or not. Also, we had planned to count how many word problem subjects generated an equation, and, of those, how many generated the start-unknown equation, the result-unknown equation, or some other form. Contrary to our expectation, among those who generated an equation, only a result-unknown equation was generated. However, as will be discussed further below, some of the word problem subjects who generated a result-unknown equation, generated the same form that we provided to our result-unknown subjects, $64 + 3(20.50) = X$, but some generated $3(20.50) + 64 = X$.

Performance as a function of condition. In the word problem condition, 95% of the participants (19 out of 20) were correct, compared to only 37% of the start-unknown participants (7 out of 19) and 47% of the result-unknown participants (9 out of 19), $X2$ (2, n=58) = 15.76, p = .0004. The word problem performance was superior to both the start-unknown performance, $X2$ (1, n=39) = 14.83, p = .0001 and the result-unknown performance $X2$ (1, n=39) = 10.92, p = .001. There was no difference in performance between the start-unknown and result-unknown conditions, *Fisher exact test (two tailed) p* = 1.0.

Equations generated. In the word problem condition, exactly half of the 20 participants produced no equation, but 9 out of those 10 correctly solved the problem. Among the other 10 subjects, who did set up an equation, 6 set up a result-unknown equation by writing $64 + 3(20.50) = X$. However, 4 also set up a result-unknown equation, but wrote it as $3(20.50) + 64 = X$. All of the participants who generated an equation solved it correctly.

Types of errors. We examined the types of errors made by all the subjects. Here, we report those types of errors by condition.

Word problem condition. In the word problem condition 10 of the 20 subjects failed to generate any equation. Of those, 9 out of 10 were nevertheless able to solve the word problem. Their work typically showed that they multiplied 3 times $20.50 to get $61.50, and then added $64 to get $125.50. In other words, they used the semantic support to follow the same steps that they would have followed in solving the equation. The one subject who failed, performed those same steps, but got $91.50 for the multiplication step.

Among the other 10 subjects, who did set up an equation, none of them made any errors in following the order of operations, doing the arithmetic, or any other type of error, regardless of which form of the result-unknown equation they wrote.

Start-unknown condition. Among the 19 start-unknown subjects, who were given $(X - 64) \div 3 = 20.50$, only 7 solved the equation correctly. Of those, 6 followed the usual order of operations. One subject used an unorthodox method, but did so correctly. This subject first divided both X and 64 by 3 to obtain X/3 and 21.33. The subject only carried the 21.33 to the hundredth place, even though the 3's should actually repeat, or 1/3 should be added. This rounding led to a small, but insignificant, error in the final result. This subject went on to add 21.33 to both sides, obtaining X/3 = 41.83 (again, carrying the 41.83 to only the hundredth place). Finally, the subject multiplied both sides by 3 to get X = 125.49. Given that this was only a rounding error, we scored the subject as correct.

Among the 12 subjects who were incorrect on $(X - 64) \div 3 = 20.50$, 7 made the same error. They began by undoing the first operation that they came to when reading left to right. They added 64 to both sides of the equation, rather than multiplying both sides by 3. Other errors were incorrect arithmetic (2 subjects), dividing both sides by 3 instead of multiplying (2 subjects), and failure to treat the (X - 64) as if it were a single term. Heffernan (2001) noted that failing to treat terms in parentheses as a single term is a common source of difficulty in correctly producing the "grammar of algebra".

Result-unknown condition. Among the 19 result-unknown subjects, who were given the $64 + 3(20.50) = X$ equation, 9 solved the equation correctly, showing that they followed the correct order of operations, and did the arithmetic correctly. But the other 10 failed to solve the equation. Of those 10, one made a decimal placement error when adding 64 to 61.50 (treating 64 as .64). However, another 8 subjects failed to follow the order of operations. They, similar to the 7 subjects noted in the start-unknown condition, began by doing the first operation that they came to when reading the equation from left to right. They incorrectly added 64 and 3 to get 67, and then attempted to multiply 67 times 20.50. One subject inexplicably simply multiplied 7 times 20.50.

Discussion

Our skepticism regarding the relative ease of the word problem compared to the relative difficulty of the start-unknown equation was not supported. We replicated the high performance on the word problem found by Koedinger et al. (2008). Indeed , we not only replicated the finding that solving the word problem was easier than solving the start-unknown equation alone, but also found that solving the word problem was easier than solving the result-unknown equation alone.

Our intuition that at least some people in the word problem condition would generate the result-unknown equation was correct. In fact, all of those who generated any equation (half the subjects) generated a result-unknown form. This lends support to our contention that a result-unknown form of the equation is more familiar to people than a start-unknown form. Nevertheless, the perfect performance of the word problem solvers who used the result-unknown equation compared to the 47% performance of those who received that equation alone suggests that, contrary to our intuition, in Koedinger et al.'s (2008) study, it was the semantic support afforded by the word problem, rather than the relatively greater unfamiliarity of the start-unknown equation, that led to their result.

However, there was also an unexpected finding. We found it interesting that, although all the word problem subjects who generated an equation generated a result-unknown form, some of them generated a form, $3(20.50) + 64 = X$, that would lead to following the correct order of operations if it were solved by performing operations from left to right. Furthermore, the errors among the equation only subjects described in the result section suggest that without the semantic support of the word problem, the subjects in the equation only conditions may have interpreted both the start-unknown equation and the result-unknown equation as simply a set of steps to be performed from left to right. For example, on $(X - 64) \div 3 = 20.50$, the most common error was adding 64 to both sides first, rather than multiplying both sides by 3 first.

However, there are at least two possible explanations for making the error of adding 64 to both sides first when presented with $(X - 64) \div 3 = 20.50$. One explanation is that these participants were treating the equation as simply as set of operations to be performed from left to right, just as one would read the equation. However, another possible explanation is that these students were trying to follow the correct order of operations for simplifying an equation, instead of reversing the order in which one undoes each operation algebraically. Algebra students are often given the acronym PEMDAS as a mnemonic to help them remember to simplify an expression by working first with Parentheses, then Exponents, then Multiplication and Division, and finally, Addition and Subtraction. Two of the subjects in Experiment 1 had actually written “PEMDAS” on their response sheets. However, if such students incorrectly believed that solving an equation algebraically required undoing each operation in the PEMDAS order, instead of in the reverse of that order, then they might think that the first step in solving $(X - 64) \div 3 = 20.50$ should be to undo whatever was in the parentheses by adding 64 to both sides. Indeed, one of the subjects who made that error was one of the subjects who had written “PEMDAS” on the response sheet.

For those subjects presented with $64 + 3(20.50) = X$, the most common error was adding first, rather than multiplying first. This error again could mean either of two things. Again, one explanation is that these participants were treating the equation as simply as set of operations to be performed from left to right instead of following the correct order of operations. The other possibility seems less likely. First, they would have had to have recognized that, because the unknown was already isolated on one side of the equals sign, this was a situation in which solving the equation only required simplifying the expression on the other side, rather than undoing operations. In addition, they would have had to have known the correct order of operations, but incorrectly thought that this was the situation in which they should follow them in the reverse order. If that is what they believed, then adding is what they would do first.

Given that the subjects were all college students, it is somewhat surprising that they would not know the order of operations. Nevertheless, it is possible that they either never sufficiently learned the order of operations or that they did not retain that knowledge. If that were the case, then in deciding what to do first, they may have defaulted to the same left to right order in which one reads. If, on the other hand, they knew the correct order of operations, it is possible, as explained above, that they were confused as to when to apply the PEMDAS order versus when to reverse it. We decided to conduct a second experiment to try to distinguish between the two possible explanations for the errors in Experiment 1 on the result-unknown equation, $64 + 3(20.50) = X$.

Experiment 2

In Experiment 2 we presented some subjects with $64 + 3(20.50) = X$. We will call that the incongruent form of the equation because working from left to right would lead to violating the order of operations. Other subjects were presented with $3(20.50) + 64 = X$. We will call that the congruent form, because because working from left to right would lead to following the order of operations.

If the previous errors were due to treating equations as sets of steps to be performed from left to right, then on the incongruent form we should see many subjects making the same error as the subjects in Experiment 1. That is, they would first add 64 and 3 to get 67, and then they would try to multiply 67 times 20.50, resulting in 1373.50 (assuming that they did the multiplication correctly). But, if the previous errors were due to treating equations as sets of steps to be performed from left to right, then we should see good performance on the congruent form.

On the other hand, if the previous errors were due to believing that in this situation one should reverse the order of operations, then they should make errors on both equations. On the incongruent form, they should make the same error described above. But on the congruent form, they would also try to add first. This would result in adding 20.50 to 64 to get 84.50. This would be followed by multiplying 84.50 times 3 to get 253.50. Although we did not expect to see this error, we conducted Experiment 2 to determine if it would occur.

Method

Participants. The participants were 98 college students at a 4 year undergraduate teaching university. They were recruited from a variety of seven different summer session classes. Two classes were in the Mathematics department. One was an introductory math course and the other was a fundamentals of math course. Three courses were in the Psychology department. There was an industrial/organizational course, a general psychology course, and a delinquency course. There was also a speech course and a graduate level research methods course. The subjects in Experiment 2 were all taking classes in a second summer session, whereas the subjects in Experiment 1 had come from students taking classes in the first summer session. This made it possible that some of the participants in Experiment 2 either had participated in Experiment 1, or had heard about it from other students. Therefore, we asked them to report if they had any knowledge of the prior experiment. Of the 98 participants, 8 reported having some previous knowledge, and, therefore, their data were not included in the analysis, leaving 90 participants whose data we analyzed.

Materials and procedure. The subjects were randomly assigned to either the incongruent or congruent condition. All the subjects were given one equation to solve. In the incongruent condition ($N = 45$) they received $64 + 3(20.50) = X$, whereas in the congruent condition ($N = 45$) they received $3(20.50) + 64 = X$. They were simply instructed to try to solve the equation and to show all their work. They were allowed to use a calculator if they wished, and they were not timed.

Hypothesis. We hypothesized that the errors on $64 + 3(20.50) = X$ when presented alone in Experiment 1 were due to the subjects' tendency to work from left to right, not to a confusion about when to use the order of operations and when to reverse them. Therefore, in Experiment 2, in the incongruent condition ($64 + 3(20.50) = X$) we expected that, as predicted by both hypotheses, we would see many errors where the subjects added 64 and 3 before multiplying by 20.50. In the congruent condition ($3(20.50) + 64 = X$) , we expected much better performance than in the incongruent condition. Also, we expected that when errors did occur, they would not be the error of failing to follow the order of operations. Specifically, we did not expect to see subjects adding 20.50 and 64 before multiplying by 3.

Results

Performance as a function of condition. In the congruent condition, 82% of the participants (37 out of 45) provided the correct solution to the equation, which was a marginally significant advantage compared to 67% of the incongruent participants (30 out of 45), X^2 (1, n=90) = 2.86, $p = .091$. Some participants were incorrect because they failed to follow the correct order of operations. However, other participants did follow the correct order of operations but were incorrect because of arithmetic errors. Because we were interested in the effect of the equation format specifically on following the correct order of operations, we also examined the percentages of participants who did so in each condition, even if they made an arithmetic error. In the congruent condition, 93% of the participants (42 out of 45) followed the correct order of operation, which was a significant advantage compared to only 78% of the incongruent participants (35 out of 45), X^2 (1, n=90) = 4.41, $p = .036$.

Types of errors. We examined the types of errors made by all the subjects. Here, we report those types of errors by condition.

Congruent condition. In the congruent condition, in which the participants received $3(20.50) + 64 = X$, out of the 8 participants who failed to provide a correct equation solution, we scored only 3 participants as failing to follow the correct order of operations. The first of those participants provided two answers. For the first answer, this participant did the multiplication step correctly, but then dropped the decimal point in the next step. Thus, this participant's first answer was 6150 + 64 = 6214. Had that been the only answer the participant gave, that would have been scored as only an arithmetic error, not a failure to follow the order of operations. However, the participant then gave a second solution. In this solution the participant multiplied both the 20.50 and 64 times three. The participant then again dropped the decimal point and gave 6150 + 192 = 6342 as the answer. This was not the type of error predicted by the hypothesis that subjects believed that in this situation they should reverse the order of operations. However, because there was an extra and incorrect step involved (multiplying 64 times 3), we stayed on the conservative side and scored it as not following the order of operations.

The second participant multiplied 20.50 times 3 correctly. But then the participant tried to subtract 61.50 from both sides. However, on the left side of the equals sign, the participant showed 61.5 + 64 minus 61.5 = 2.5. Apparently the participant subtracted the 61.5 from the 64 and just neglected the other 61.5. Neglecting the other 61.5 was ultimately the error that led to an incorrect answer. On the right side of the equals sign the participant produced X - 61.5. For the final step, the participant added 61.5 back to both sides. Had this participant retained the other 61.5 on the left side, adding 61.5 back to both sides would have resulted in returning the equation to a state it had been in earlier, 61.5 + 64 = X. Had that occurred, then maybe the participant would have realized that all that needed to be done was to add the 61.5 and the 64 to arrive at the correct answer, 125.5. But, because the other 61.5 had been dropped, adding 61.5 back to both sides led the participant to the answer, 64 = X. This was, again, not the type of error predicted by the hypothesis that subjects believed that in this situation they should reverse the order of operations. However, because a necessary step was omitted (retaining the other 61.5 on the left side of the equals sign), we again stayed on the conservative side and scored it as not following the order of operations.

The third participant also showed two answers. In one place the participant showed the correct order of operations and the correct answer. But in another place on the page, the participant showed some other work that led to an incorrect answer. Because we had no way to tell which answer the participant intended to give, we again decided to err on the side of interpreting the participants responses conservatively with regard to its potential to support our hypothesis and held the participant responsible for the incorrect answer. The participant began by trying to subtract 64 from both sides, but showed as the result, 3(20.50) = -64X. In other words, the critical error was to show that X minus 64 results in X being multiplied by a negative 64. Therefore, in the next step, the participant divided both sided by -64. On the right side of the equals sign, the participant showed correctly that -64X divided by -64 equals X. But on the right side, the participant showed that 61.5 divided by -64 equals .96. So, the participant failed to show that it should have been a negative .96, but that was beside the point. The critical error had already led the participant to the wrong procedure. Once again, this was not the type of error predicted by the hypothesis that subjects believed that in this situation they should reverse the order of operations. However, because a step in the order of operations had produced an incorrect result (X minus 64 does not equal -64X), we again stayed on the conservative side and scored it as not following the order of operations. Therefore, it should be noted that no subject made the error that we predicted should occur if subjects were confusing when to use the PEMDAS order of operations versus when to reverse them.

Of the 8 participants in the congruent condition who gave an incorrect answer, the other five did follow the correct order of operations. All of those subjects got an incorrect result when they multiplied 3 times 20.50. Two of them additionally made place value errors, adding 64 to their incorrect result as if it were .64.

Incongruent condition. In the incongruent condition, in which the participants received *64 + 3(20.50) = X*, out of the 15 participants who failed to provide a correct equation solution, we scored 10 participants as failing to follow the correct order of operations. All 10 participants made the same error. They added 64 and 3 to get 67 and then tried to multiply 67 times 20.50. Of those 10, only 5 multiplied correctly to reach 1373.50. The other 5 made a variety of additional errors in their multiplication.

The other 5 participants did follow the correct order of operations. All of those subjects got an incorrect result when they multiplied 3 times 20.50. Two of them got 63.50, two got 60.50, and one got 64.50.

Discussion

In Experiment 2, subjects in the congruent condition correctly solved their equation more often than the subjects in the incongruent condition, although this difference would only have been statistically significant at the .05 alpha level in a one tailed, not a two tailed, test. However, the congruent subjects followed the order of operations significantly more often than did the incongruent subjects. In fact, no subjects in the congruent condition actually reversed the PEMDAS order of operations and tried to add 20.50 to 64 before multiplying by 3. Thus, these results suggest that the reason why the subjects in the equation only conditions of Experiment 1 often failed to follow the correct order of operations was that they thought that they should just work from left to right, not that they thought simplifying an expression required them to reverse the usual PEMDAS order of operations.

General Discussion

The first of these two experiments confirmed that students can benefit from the semantic support provided by a simple word problem. Specifically, it showed that such a word problem is easier for students than a corresponding equation regardless of whether the equation is presented in the start-unknown format, *(X - 64) ÷ 3 = 20.50*, or the result-unknown format, *64 + 3(20.50) = X*. However, the second experiment showed that students can also benefit from having a result-unknown equation presented to them in a format that is congruent with the order of operations. That is, students tend to work on the equation from left to right, and, therefore, they perform better on an equation, such as *3(20.50) + 64 = X*, in which working from left to right results in following the correct order of operations, than on an equation such as *64 + 3(20.50) = X*, in which working from left to right results in violating the order of operations.

Theoretical Implications

Koedinger et al. (2008) proposed a representation-complexity trade-off, in which, for single reference problems, such as *(X - 64) ÷ 3 = 20.50*, a word problem format facilitated problem solving performance compared to the corresponding equation, whereas for double reference problems, such as $x + (x + 6) = 38$), the opposite was true. Given that it has previously been generally held that word problems are harder than equations, it was the facilitation of word problem performance that was the part of the trade off that was an innovation. The trade-off results from the relative unfamiliarity of the equation compared to the semantic support provided by the word problem. In Koedinger et al.'s formulation, equations have the advantage of providing working memory support and efficiency by being externalizable and concise. However, because of the simplicity of a single reference problem, those advantages do not do enough to outweigh the semantic support advantage of the word problem over the unfamiliarity disadvantage of the equation.

Our original hypothesis was that the unfamiliarity disadvantage of the equation stemmed from the placement of the unknown. That is, we thought that a start-unknown equation was more unfamiliar than a result-unknown equation. Our first experiment showed that to be wrong. However, we then noticed that the equation that Koedinger et al. (2008) used was not only a start-unknown equation, it was also incongruent with the order of operations. In our first experiment, we had compared a start-unknown to a result-unknown equation, while holding congruence constant. That is, both of the equations we used in the first experiment were incongruent. In our second experiment we compared a congruent to an incongruent equation while holding the position of the unknown constant. That is, both of the equations were result-unknown. That experiment showed that, at least for result-unknown equations, a congruent equation is easier than an incongruent one. Therefore, the unfamiliarity disadvantage for the equation that enabled Koedinger et al. (2008) to uncover the new finding that, for single reference problems, a word problem format is better than an equation format, may have stemmed from the equation being incongruent.

Future Research

According to the analysis above, the results of our second experiment imply that if, in our first experiment, we had compared performance on the word problem to performance on a start-unknown equation alone in the incongruent form, $(X - 64) \div 3 = 20.50$, and to performance on a result-unknown equation alone in the congruent form, $3(20.50) + 64 = X$, then we would have seen that performance on the word problem would have been better than on the former equation, but *not* on the latter. If we had done that, however, we would have been confounding whether the equation was start-unknown versus result-unknown with whether it was incongruent or congruent. In order to examine those two factors independently in the same experiment, we would have to compare performance on the word problem to equations that varied independently on the two dimensions of where the unknown was placed, and whether or not it was congruent. An example of four equations that would meet these requirements is shown in Table 1. Those equations are based on the equations used in Koedinger et al. (2008). However we changed the 64 to 63 so that the congruent/start-unknown equation would contain 21, rather than 21.33 with a repeating decimal.

Table 1
Example of equations that could be used to examine the difficulty of equations, relative to a word problem, as a function of the placement of the unknown and congruence with the order of operations.

	Unknown	
	Start-Unknown	Result-Unknown
Congruent	$-21 + \frac{X}{3} = 20.50$	$3(20.50) + 63 = X$
Incongruent	$(X - 63) \div 3 = 20.50$	$63 + 3(20.50) = X$

Practical Implications for Instruction

If the comparison described above were to show that the simple word problem was easier than the incongruent equations, but not the congruent equations, then that raises the possibility that, just as simple word problems might better be taught before corresponding equations in their incongruent format, it might also be better to teach congruent equations before incongruent equations. In fact, teaching equations in that order and explicitly differentiating between when equations were in a congruent versus an incongruent format might help students to better learn, retain, and later apply the correct order of operations.

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