Chapter 0
Reasoning

§ 0.1 REASONING.

Throughout the text a concerted effort will be made to encourage the student to discern the difference in types of reasoning and the importance of reasoning in mathematics. Broadly speaking, there are two types of reasoning central to life, to science, and especially to mathematics. One reasons by attempting to draw generalisations from patterns, from observed outcomes of events, from an example, or a series of examples. The type of reasoning used in this manner is called **inductive reasoning**. It is used to hypothesise, conjecture, and form a basis for many theories that are fundamental building blocks to our understanding of our lives, our world, and our universe. Regrettably, there is a basic flaw in this form of reasoning. The existence of a single instance of a contradictory result, a contradictory event, or such that a pattern that seems to be really isn't destroys the most parsimonious or elegant theory.

For example, consider 1, 2, 4, 8, 16, . . . We have seen such in our secondary or primary school studies and some believe that the ellipsis (the three dots) *causes* there to be a pattern which continues. There is no such cause inherent in the commas, the ellipsis, nor in their combination. Consider 1.4142135623730950488016887242097 … which is the decimal expansion for $\sqrt{2}$. We shall see (indeed prove in Math 255, Set Theory [the subsequent course to this course]) that $\sqrt{2}$ is an irrational number so the decimal expansion is non-repeating hence does not have a pattern! So the ellipsis only 'suggests' patterns where none such exist.

Mathematicians use inductive reasoning to formulate ideas, conjectures, and hypotheses. Such reasoning forms the foundation that one uses in order to progress. However, it does not guarantee that is right. In order to establish the veracity of a claim one must prove it. So, mathematics is built on a foundation, as we shall see, where there are primary or atomic assumptions made and then the mathematician deduces a result. The manner of reasoning so outlined is called **deductive reasoning**. In deductive reasoning the emphasis is not on events or particulars but on the general abstract thought process so that from the general property whilst applying the laws of logic one can draw necessary conclusions. It is not the grey terminology that the foundationalist seeks but the concrete understanding of a mathematical system or principle.

For example suppose one was told that for the counting numbers that we shall generally call, 'n,' (1, 2, and so forth) we are allowed to assume that we have the property that *f*(n) = $2^{(n-1)}$. From this assumption we can deduce that *f*(1) = $2^{(1-1)} = 2^0 = 1$. We can deduce that *f*(2) = $2^{(2-1)} = 2^1 = 2$. We can deduce that *f*(3) = $2^{(3-1)} = 2^2 = 4$; etcetera ad nausea. So, we have formulated from the general to the specific that we have a sequence (which many students have studied in secondary school, in the Analysis sequence, but which will be rigorously defined in Mathematics 361, Real Analysis) so that GIVEN *f*(n) = $2^{(n-1)}$ where n is a natural number (defined in chapter 2) we have the sequence so that we have 1, 2, 4, 8, . . . where the ellipsis indicates the pattern continues <u>because of the definition</u> *f*(n) = $2^{(n-1)}$ where n is a natural number rather than in spite of said definition or that because a pattern *seems* to be makes it be (which is utter balderdash).

Many mathematics in the late twentieth century began studying non-rigorous or non-closed systems such that they are not deterministic as we have indicated previously. Such open or dynamic systems are *ripe* for study and are in every way, shape, and form *worthy* of attention.

However, we shall approach our study of mathematics from the deterministic or stable vantage point and realise that once we understand the concrete, we can then study dynamic (dynamical systems, differential equations, etc.) or stochastic (statistics, measurement analysis, etc.).

Let us consider the following problem which is quite amusing. It is a game as are many games we play as youths where the object of the game is to solve the puzzle and come up with the correct solution.

Example 0.1.1: The Circle Problem

Let C be a circle with radius one centred at (0,0).

Set 1. Let A_1 and A_2 be points on the circle. Call them vertices. Connect all vertices with a chord (in this case A_1 and A_2 are connected by a chord). Consider the interior of the circle. Let region be defined as an interior part of the circle such that a set of chords separates it from other interior parts of the circle. The chord itself is a 'bound' and as such is neither considered a part of nor a region. Count the number of points (2), the number of chords, (1), and the number of regions (2).

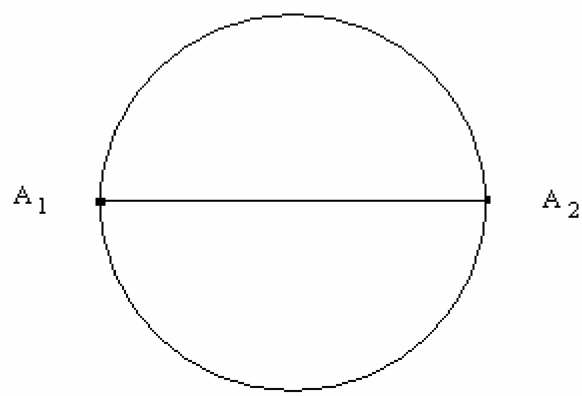

Figure 0.1.1

Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

Set 2. Let A_1, A_2, and A_3 be points on the circle. Call them vertices. Connect all vertices with a chord (in this case A_1 and A_2 are connected by a chord, A_1 and A_3 are connected by a chord, and A_3 and A_2 are connected by a chord). Consider the interior of the circle. Let region be defined as an interior part of the circle such that a set of chords separates it from other interior parts of the circle. Count the number of points (3), the number of chords, (3), and the number of regions (4).

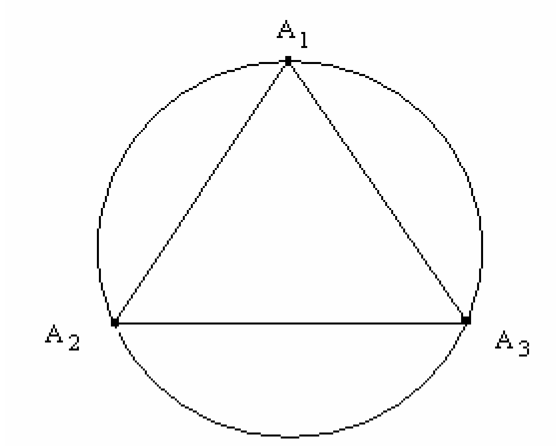

Figure 0.1.2

Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

Set 3. Let A_1, A_2, A_3, and A_4 be points on the circle. Call them vertices. Connect all vertices with a chord (in this case all vertices are connected by a chord. Consider the interior of the circle. Let region be defined as an interior part of the circle such that a set of chords separates it from other interior parts of the circle. Count the number of points (4), the number of chords, (6), and the number of regions (8).

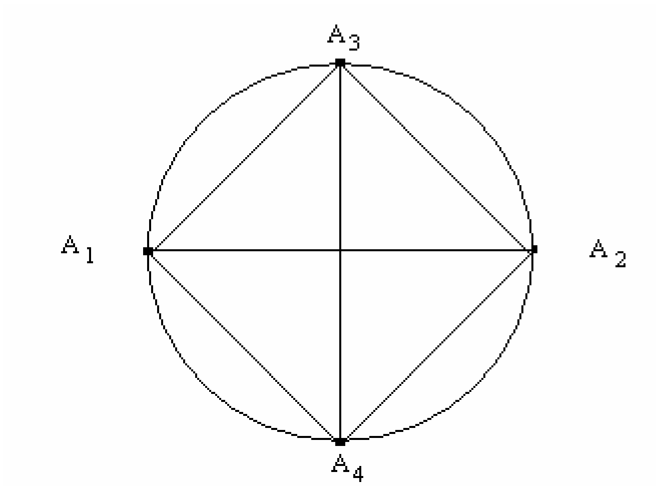

Figure 0.1.3

Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

<u>Exercise (see exercise set)</u>: Do this for set 5, 6, 7, and 8. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

Have you determined any flaw in your reasoning? If so, what flaw; if not, why?

Let us consider the following problem which is also quite illuminating. It is a problem that one could easily find in any high school algebra book. The object of the exercise is of course to solve the puzzle and construct the correct solution.

Example 0.1.2: The Polynomial Problem

Consider $f(n) = n^2 + n + 41 \ \forall \ n \in \mathbb{N}$ (usually in high school it would be stated as "consider $f(n) = n^2 + n + 41$ for each n being 1, 2, 3, 4, and so forth [never ending]").

Set 1. Consider $f(1)$. It is $1^2 + 1 + 41 = 43$. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 2. Consider $f(2)$. It is $2^2 + 2 + 41 = 4 + 2 + 41 = 47$. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 3. Consider $f(3)$. It is $3^2 + 3 + 41 = 9 + 3 + 41 = 53$. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 4. Consider $f(4)$. It is $4^2 + 4 + 41 = 16 + 4 + 41 = 61$. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 5. Consider $f(5)$. It is $5^2 + 5 + 41 = 25 + 5 + 41 = 71$. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Exercise (see exercise set): Do this for set 6, 7, 8, 9, 10, and 11. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

Have you determined any flaw in your reasoning? If so, what flaw; if not, why?

Each of the examples are rather facile once on understands the basic problem. Nonetheless, note that some of the patterns one might opine exists either exists or does not. There is no guarantee as to the veracity of the claim without some manner of proof (which is what we will be studying later).

Example 0.1.3: The Hard Polynomial Problem

Consider $f(n) = 991n^2 + 1 \; \forall \; n \in \mathbb{N}$

Set 1. Consider $f(1)$. It is 992. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 2. Consider $f(2)$. It is 3,965. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 3. Consider $f(3)$. It is 8,920. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 4. Consider $f(4)$. It is 15,857. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Set 5. Consider $f(5)$. It is 24,776. What kind of natural number is it? Hypothesize as to the general rule for $f(n)$. How confident are you with your prediction? Do you think your hypothesis is true or false? Why?

Exercise (see exercise set): Do this for set 6, 7, 8, 9, 10, and 11. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

§ 0.1 EXERCISES.

In this exercise set (and this set only) the solutions are provided on the following page.

1. Do the circle problem for set 5. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

2. Do the circle problem for set 6, 7, and 8. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

3. Do the polynomial problem for set 6, 7, 8, 9, 10, and 11. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

4. Do the polynomial problem for set 15, 20, 25, 30, 35, 40, and 45. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

5. Do the hard polynomial problem for set 6, 7, 8, 9, 10, and 11. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

6. Do the polynomial problem for set 100, 200, 1,000, 10,000, 1,000,000, and 1,000,000,000. Hypothesize as to the general number of points, chords, and regions. How confident are you with your predictions? Do you think your hypotheses are true or false? Why?

Hint 1. The Circle Problem
Consider $n = 6$

Hint 2. The Polynomial Problem
Consider $f(40)$

Solution 1[1]: For n = 6 the conjecture about regions fails for $2^5 = 32$, but one will get 30 or 31 depending on the placement of the vertices (31 if the vertices are not equidistant around the circle, 30 if they are).

Solution 2[2]: For n = 40 the conjecture $f(n)$ is prime fails since $f(40) = 41^2$.

Solution 3[3]: For n = 12,055,735,790,331,359,447,442,538,767 the conjecture $f(n)$ is not a perfect square fails since $f(12{,}055{,}735{,}790{,}331{,}359{,}447{,}442{,}538{,}767) = 10^{28}$.

The point of the exercise id to realise that inductive patterns are not generally true even if there are many examples with which to point to as evidence of a pattern. It is the case that many people get tired with checking each and every case (with the natural numbers on cannot check each and every case) so one skips and jumps about checking a certain number of cases (a heuristic – a rule of thumbs - by which one satisfies oneself with the pattern) and then claims 'truth.' For truth to exist for a human there must be a proof to accompany the claim. Proof is a central part of mathematics and will occupy our attention for the remainder of the course.

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[1] I cannot remember where I found this problem.

[2] Schumaker, C. *Chapter Zero,* (Reading, MA: Addison-Wesley, 1997), page 66.

[3] Schumaker, C. *Chapter Zero,* (Reading, MA: Addison-Wesley, 1997), page 66.