Introductory Statement

A liberal arts education is founded upon the principle that one should learn about the principle achievements of the human race. Learning about the achievements of the human race involves reading, opining, reflecting, and doing. It is in the doing that one begins to experience some of the richest and most satisfying components of the liberal arts tradition. This is because reading or hearing about principles is all well and good, but it does not enable a person to go and discover, create, or invent and add to the canon of human knowledge. Indeed, the mere recitation of facts and principles of the canon is more an exercise for a PBS documentary than for a college course. Nonetheless, the appreciation of that which has come before is a part of the college experience.

A modern mathematics education is founded upon the same principles. It is concerned with a particular aspect of the canon: the discernment between the veracity or lack thereof of claims. A sagacious man is one who is willing to entertain the possibility that his ideas are wrong. He is brave enough to investigate this possibility with an objective framework which outlines a scheme to arrive at a conclusion such that the claim under investigation is shown to be true, false, or non determinable based on the knowledge available.

Thus, one particular component of the canon that is central to mathematics is the search for truth. Suppose a person tells you, "the sky is green." You would likely say that that was not true based on the empirical evidence that the sky is not green, but blue or grey on most occasions. However, can it ever be green? Such claims are not what mathematics is about. Mathematics is not so concerned with temporal questions, questions of perceptions, questions of opinion, but with questions which transcend such a temporal realm. That is not to say that temporal questions are not asked in mathematics, but at this beginning level we will concern ourselves with more general claims.

Indeed mathematics is many things. It is a language that formalises abstract concepts and thoughts. It is a collection of knowledge about relations, measurements, figures, processes, objects, quantities, etc. It is a method of knowing. I do not think there is one definition which does justice to the subject, thus I shall not burden the reader with rudimentary mental reflections on what it is or is not. I will simply allow for the subject to exist and assume the reader is interested in unlocking its "secrets" and in so doing understand the subject better than when he entered this course.

Some of the principles of mathematics that we will study in this course will be nuanced, but all of them will have as their nature a characteristic such that they will be useful in later studies, they will be applicable in multiple situations, and they will be provocative. We will study some of the major findings of the past 150 years or so and will endeavour to do so such that we not only discuss the findings but discover, create, or invent some of the findings ourselves. By doing such we will better understand the principles of mathematics and truly *know* mathematics rather than *witness* mathematics. So much of the educational experience of students today is founded upon a climate of witnessing: use of a computer or calculator, listening to a lecture, watching a video, etc.
I do not accept this as a real liberal arts educational experience. Watching someone do mathematics is not *knowing* mathematics just as watching Andy Roddick play tennis does not imply one *is* a tennis player. This analogy illustrates what is ahead for the student enrolled in this course. You will be expected to do mathematics, not watch mathematics

being done. You will be expected to be conversant in the language of mathematics, not have a translator by your side to punch in words and get definitions. You will be expected to be comfortable with the processes and methods illustrated and discussed and the only way I know to be capable of doing such is by doing homework.

Real learning takes place not in the classroom, but in other settings colloquially referred to as "at home;" hence, homework. Homework for this course requires memorisation of some symbols, axioms, definitions, lemmas, theorems, and corollaries (oft called rote), involves working multiple problems (oft called drill exercises to reinforce key methods), and tackling complex problems. The time spent in class is designated to come together as a community of scholars and reflect on what we understand, revise and extend that which we may not be truly clear on, and build on that which we have mastered.

A solution to a problem, a proof of the veracity of a claim or a counter-argument is correct not by the mindless memorisation of an algorithm, the copying of a solution from a book or notes, by punching buttons on a calculator, or by authority of the instructor but because it is reasoned correctly. We shall learn to reason properly, to write out results cogently, and to critically review arguments which claim truth.

If this is not to one's liking, then perhaps the college experience is not for him or at the very least perhaps mathematics is not for him. To such a student my best advise is seek out other avenues of learning or expression because it is a waste of your time, it is a waste of my time, and it is a waste of other students time to be in a class from which you receive nothing, are miserable in, and wish not to be enrolled. Life (and mathematics) is a banquet, and if you don't like the menu, leave the restaurant and find a cuisine more appealing to your pallet!

This text is not merely a collection of exercises such that it is referenced only when attempting homework. It should be read before class commences. It should be read after the class discussion for reinforcement. It should be referenced when doing homework. Many mathematics texts are written in such an obtuse manner that reading them can be a dreadful experience. I hope such is not the case with this text. However, in my career, I have found there were times that some of the most difficult texts were better than no text, there is no rule that states one cannot reference another text, and if you find a problem with the exposition, then by all means please tell me - - this is a work in progress. However, please note that I intentionally have attempted to write this text on a college level, I am not interested in "dumbing down" the material or in any way "talking down" to the student, and am not amenable to revising this tenet.

The justification for the existence of this text is complex, but can be summed up by noting that there are a number of introductory texts available on the market which give an overview of advanced mathematics and are intended to be exercises in the transition for a student to upper level mathematics. I opine that many of them are quite acceptable, but do not contain all of the material that we will discuss in the semestre, many of them are written on such an elementary level as to be laughable and are not fitting to be part of our course, and many are written at too advanced a level such that the prerequisite knowledge assumed is more advanced than can be justified in the United States of the twenty-first century. So, like the bear's porridge, I hope this text is "just right."[1] I assume that a student has learned elementary school arithmetic: addition, subtraction, multiplication, division, manipulation of fractions, decimals, and has been exposed to bases other than ten. I assume that a student can properly manipulate algebraic expressions, can prove basic claims in geometry, has an understanding of trigonometry, functions, graphs, conic sections, natural, prime, rational, irrational, real, and complex numbers. I assume a student can solve linear, quadratic compound, absolute, and higher order equations and inequalities. Nonetheless, if a subject is "sketchy," if the memory of such is difficult to recall, or if such was not discussed in a student's primary or secondary education, then it is the student's responsibility to remediate him; he should review said material as soon as the problem has surfaced.

Mathematics is quite literally a subject that is built like a building. It rests upon certain fundamentals - - the foundation, and can only be mastered if said foundation is strong. There are many short cuts that one can follow that work for a day, a week, a year, or longer but eventually come back to "haunt" the individual because the required skill or understanding that is central to concept that the short cut bypassed is not understood by the student; hence, the short cut was not a favour but a burden in the long run. So it is with learning mathematics, better to take time now and understand than try to rush through homework and not really understand but to get it done. Mathematics is sequenced in a way such that subsequent discussion is predicated upon previous topics.

So too in this course we shall see that the course is laid out sequentially - - each topic follows (most often) from previous. Further, mathematics is not a subject that is joined by unary ideas so that each is unique and independent of other topics. Hence, a refinement such that one topic rests with discussion of others is not possible. So even though the chapters are designated one, two, three, etc. let us consider for example material from set theory and logic. The two are not independent, so elementary set theory concepts and logic concepts will be introduced simultaneously. Further refinement and expansion will be relegated to individual chapters or sections.

Please note that terms such as is, are, must, right, and wrong are used liberally throughout the text. This is due to the nature of pure mathematics. Assuming a set of axioms and using logic we can **deduce** certain truths (based on the axioms). So when you read a statement such as, "$3 \geq 2$ by the law of addition - - since the statement $3 > 2$ is true it *must* follow that $3 > 2 \vee 3 = 2$," understand that is predicated upon the axioms (of the reals).

[1] As in class lectures, my humour will infect this text (apologies). It is the case that though try as I might, I simply cannot resist a corny joke or two, but with the help of my assistant, I will try to minimise the instances of the jokes and will ensure that they in no way compromise the pedagogical value of the material.