HANDOUT 4
SOME AXIOMS
MATH 545 STATISTICAL INFERENCE AND SAMPLING THEORY

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Recall from Set Theory that every set, A, is a subset of some well defined universe, *U*.
So, for probability theory we rename the universe as a sample space [the space from whence a sample may be chosen] and an arbitrary set is called an event.

The Axioms of Probability

Let S denote the sample space, E, E_i, F, etc. events and the notation Pr(•) the probability of whatever.

<u>Axiom 1</u> S is the space $\Rightarrow$ Pr(S) = 1

<u>Axiom 2</u> E is an event $\Rightarrow 0 \le \Pr(E) \le 1$

<u>Axiom 3</u> Let I be an index set. The collection $\{E_i\}_{i \in I}$ being mutually exclusive $\Rightarrow \Pr(\bigcup_{i \in I} E_i) = \sum_{i \in I} \Pr(E_i)$

<u>Corollary 1</u> E is an event $\Rightarrow \Pr(E^C) = 1 - \Pr(E)$

<u>Corollary 2</u> E and F are events $\ni E \subseteq F \Rightarrow \Pr(E) \le \Pr(F)$

<u>Corollary 3</u> Let I be an index set. The collection $\{E_i\}_{i \in I}$ being mutually exclusive and exhaustive $\Rightarrow \Pr(\bigcup_{i \in I} E_i) = \sum_{i \in I} \Pr(E_i) = 1$

<u>Theorem 1</u> E and F are events. $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$

<u>Corollary 4</u> E and F are events. $E = F \Rightarrow \Pr(E) = \Pr(F)$

Note: the axioms of probability are one of the shortest lists I can recall for an area of mathematics.
Note some of the other axioms lists for other classes.

The Field Axioms of $\mathbb{R}$

Axiom 1 (closure of addition): $\forall$ x, y $\in \mathbb{R}$, x + y $\in \mathbb{R}$ and (x = w $\wedge$ y = v) $\Rightarrow$ (x + y = w + v)

Axiom 2 (commutative of addition): $\forall$ x, y $\in$ \\$\mathbb{R}$, x + y = y + x.

Axiom 3 (associative of addition): $\forall$ x, y, z $\in \mathbb{R}$, (x + y) + z = x + (y + z)

Axiom 4 (existence of identity of addition): $\exists$ a unique number 0 $\ni$ x + 0 = x $\forall$ x $\in \mathbb{R}$

Axiom 5 (existence of additive inverse): $\forall$ x $\in \mathbb{R}$ $\exists$ a unique number -x $\ni$ x + (-x) = 0

Axiom 6 (closure of multiplication): $\forall$ x, y $\in \mathbb{R}$, x $\cdot$ y $\in \mathbb{R}$ and (x = w $\wedge$ y = v) $\Rightarrow$ (x $\cdot$ y = w $\cdot$ v)

Axiom 7 (commutative of multiplication): $\forall$ x, y $\in \mathbb{R}$, x $\cdot$ y = y $\cdot$ x.

Axiom 8 (associative of multiplication): $\forall$ x, y, z $\in \mathbb{R}$, (x $\cdot$ y) $\cdot$ z = x $\cdot$ (y $\cdot$ z)

Axiom 9 (existence of identity of multiplication): $\exists$ a unique number 1 $\ni$ x $\cdot$ 1 = x $\forall$ x $\in \mathbb{R}$ (1 $\neq$ 0).

Axiom 10 (existence of multiplicative inverse): $\forall$ x $\in \mathbb{R}$ $\ni$ x $\neq$ 0 $\exists$ a unique number x^{-1} $\ni$ x $\cdot$ (x^{-1}) = 1

Axiom 11 (distributive of multiplication over addition): $\forall$ x, y, z $\in \mathbb{R}$, x $\cdot$ (y + z) = (x $\cdot$ y) + (x$\cdot$ z)

The Order Axioms of $\mathbb{R}$

Axiom 12 (trichotomy): $\forall$ x, y $\in \mathbb{R}$, exactly one of the following relationships exists between x and y :

x < y, x = y, $\vee$ x > y. [(x < y) exor (x = y) exor (x > y)]

Axiom 13 (transitive): $\forall$ x, y, z $\in \mathbb{R}$, [(x < y) $\wedge$ (y < z)] $\Rightarrow$ (x < z)

Axiom 14 (preservation of order under addition): $\forall$ x, y, z $\in \mathbb{R}$, (x < y) $\Rightarrow$ (x + z < y + z)

Axiom 15 (preservation of order for positive multiplier): $\forall$ x, y $\in \mathbb{R}$, [(x < y) $\wedge$ (0 < z)] $\Rightarrow$ (x · z < y · z)

The Axioms of Set Theory

Axiom 1 (The Axiom of Extension) Two sets are equal iff they have the same elements.

Axiom 2 (The Axiom of Null) There exists a set with no elements, call it $\varnothing$.

Axiom 3 (The Axiom of Pairing) Given any sets A and B, there exists a set C whose elements are A and B.

Axiom 4 (The Axiom of Union) Given any set A, the union of all elements in A is a set.

Axiom 5 (The Axiom of Power Set) Given any set A, there exists a set B consisting of all the subsets of A.

Axiom 6 (The Axiom of Separation) Given any set A and a sentence p(a) that is a statement for all $a \in A$, then there exists a set B = { $a \in A$: p(a) is true}.

Axiom 7 (The Axiom of Replacement) Given any set A and a function *f* defined on A, the image *f*(A) is a set.

Axiom 8 (The Axiom of Infinity) There exists a set A such that $\varnothing \in A$, and whenever $a \in A$, it follows that $a \cup \{a\} \in A$.

Axiom 9 (The Axiom of Regularity) Given any non-empty set A, there exists an $a \in A$ such that $a \cap A = \varnothing$.

Axiom10 (The Axiom of Choice) Given any non-empty set A whose members are pair-wise disjoint non-empty sets, there exists a set B consisting of exactly one element taken from each set belonging to A.

Problem Set I

Some claims to prove or disprove (M.S.Ed. Math majors)

Claim 1 Let S be a well defined sample space, and let E and F be events ∍ $E \subset F \Rightarrow \Pr(E) < \Pr(F)$

Claim 2 Let S be a well defined sample space, $E = \varnothing \Rightarrow \Pr(E) = 0$

Claim 3 Let S be a well defined sample space, $\Pr(E) = 0 \Rightarrow E = \varnothing$

Problem Set II

This is an exercise set on the naïve or intuitive idea of probability.

Incorporate both your intuitive idea of probability of an event, E, from a finite sample space, S, being $\frac{|E|}{|S|}$;

or from an infinite one dimensional sample space , S, being $\frac{\ell(E)}{\ell(S)}$ (and you can generalise this idea for higher dimensions).

Also keep in mind the axioms of probability and the idea of a well-defined sample space, S, and an event E we have a function, p, from $\Omega \subseteq \mathcal{P}(S) \ni p\colon \Omega \longrightarrow [0, 1]$.

1. An urn contains 4 red balls, 3 green, and 5 black balls. A ball is picked. Find the probability of not picking a red ball (on one try).
2. An urn contains 4 red balls, 3 green, and 5 black balls. A ball is picked. Find the probability of picking a red or green ball (on one try).
3. An urn contains 4 red balls, 3 green, and 5 black balls. A ball is picked. Find the probability of picking a black ball (on one try).
4. An urn contains 4 red balls, 3 green, and 5 black balls. A ball is picked. Find the probability of picking a red and green ball (on one try).
5. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, pudding). You eat a meal consisting of a salad, then an entrée, then a dessert. Find the probability that a randomly chosen meal consists of a Caesar salad, pork entrée, and then cake).
6. Suppose you have 3 choices for a salad (Caesar, Tossed, Spinach), 5 choices for an entrée (fish, pork, steak, chicken, duck), and 4 choices for a dessert (ice cream, cake, pie, pudding). You eat a meal consisting of a salad, then an entrée, then a dessert. Find the probability that a randomly chosen meal consists of a Caesar salad, not a pork entrée, and then cake or pie).
7. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing all heads (exactly four heads).
8. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing a head, then a tail, then a head, then a head (in that order).
9. A fair coin is tossed 4 times and the sequence of heads and tails is observed. Find the probability of tossing 3 heads and a tail (any order).
10. Consider a pair of fair six-sided (standard) dice. You roll the dice. Find the probability of rolling a seven.
11. Consider a pair of fair six-sided dice. You roll the dice. Find the probability of rolling a seven and an eleven.

12. Consider a pair of fair six-sided dice. You roll the dice. Find the probability of rolling a seven or an eleven.
13. Consider a pair of fair six-sided dice. You roll the dice. Find the probability of rolling a seven or eleven.
14. Consider a pair of fair eight-sided dice. You roll the dice. Find the probability of rolling a seven.
15. Consider an eight-sided die. You roll the die. Find the probability of rolling a seven.
16. Consider an eight-sided die. You roll the die. Find the probability of rolling an eleven.

Problem Set III

This is an exercise set on conditional probability

1. Suppose S be a well defined sample space with $A \wedge B$ events and we know that $\Pr(A) = 0.75$, $\Pr(B) = 0.48$; and, $\Pr(A \cup B) = 0.89$.

A. Find $\Pr(A \mid B)$ B. Find $\Pr(B \mid A)$ C. Find $\Pr(A \cap B)$ D. Find $\Pr(B \cap A)$
E. Find $\Pr(A^C \mid B)$ F. Find $\Pr(B^C \mid A)$ G. Find $\Pr(A^C \cap B^C)$ H. Find $\Pr(B^C \cap B)$

2. Suppose S be a well defined sample space with $A_2 \wedge B_2$ events and we know that $\Pr(A_2) = 0.34$, $\Pr(B_2) = 0.34$; and, $\Pr(A_2 \mid B_2) = 0$

A. Find $\Pr(B_2 \mid A_2)$ B. Find $\Pr(B_2^C \mid A_2^C)$ C. Find $\Pr(A_2 \cap B_2)$ D. Find $\Pr(B_2^C)$
E. Find $\Pr(A_2^C \mid B_2)$ F. Find $\Pr(B_2^C \mid A_2^C)$ G. Find $\Pr(A_2^C \cap B_2^C)$ H. Find $\Pr(B_2^C \cap B_2)$

3. Suppose a pair of dice is tossed. Find the probability that the sum of the sides facing up is more than 5 given that you rolled a 7.

4. Suppose a pair of dice is tossed. Find the probability that you rolled a seven given the sum of the sides facing up is more than 5.

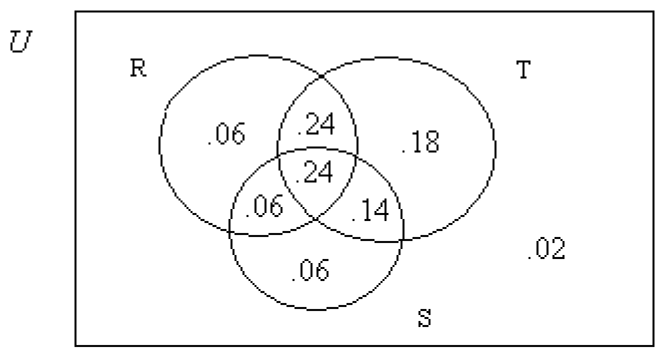
5. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls.
A. Two balls are drawn from the urn. Find the probability that both balls drawn are red.
B. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the first ball is red and the second ball is green.
C. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the first ball is green and the second ball is green.
D. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that one of the balls is red and the other is white.
E. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is red given the first ball is green.
F. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is white given the first ball is green.
G. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the second ball is green given the first ball is green.
H. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that one of the balls is red and the other is not blue.
I. Put all the balls back, shake up the urn and draw two balls (in succession) from the urn. Find the probability that the first ball isn’t red and the second ball isn’t green.

6. There is an urn. It contains 8 white, 3 red, 4 green, and 6 blue balls.
A. Three balls are drawn from the urn. Find the probability that all of the balls drawn are red.
B. Put all the balls back, shake up the urn and draw three balls from the urn. Find the probability that one of the balls is red and the other two are not white.
C. Put all the balls back, shake up the urn and draw three balls from the urn. Find the probability that the at least one of the balls is blue.
D. Put all the balls back, shake up the urn and draw three balls from the urn. Find the probability that the at least one of the balls is green.
E. Put all the balls back, shake up the urn and draw three balls from the urn. Find the probability that the at least one of the balls is yellow.

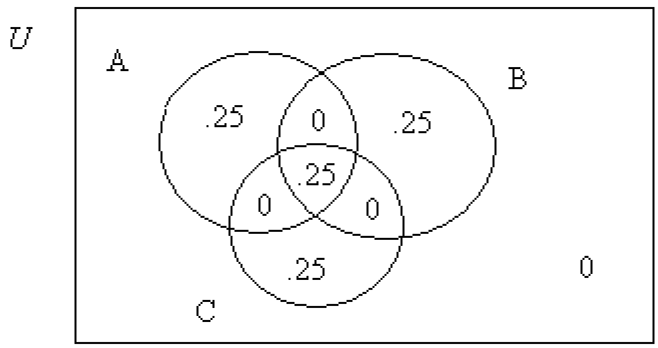
Problem Set IV

This is an exercise set on independence

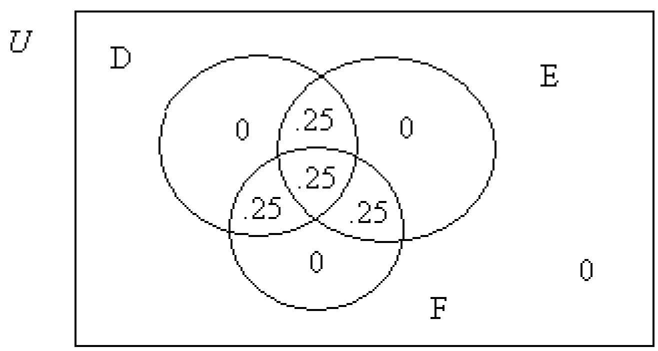
1.
Let the sample space S = U be well defined and let R, S, and T be events. With the probabilities assigned for events as given in the Venn Diagramme determine if R, S, and T are independent. Create an example for U, R, S, and T suggested by this Venn Diagramme.



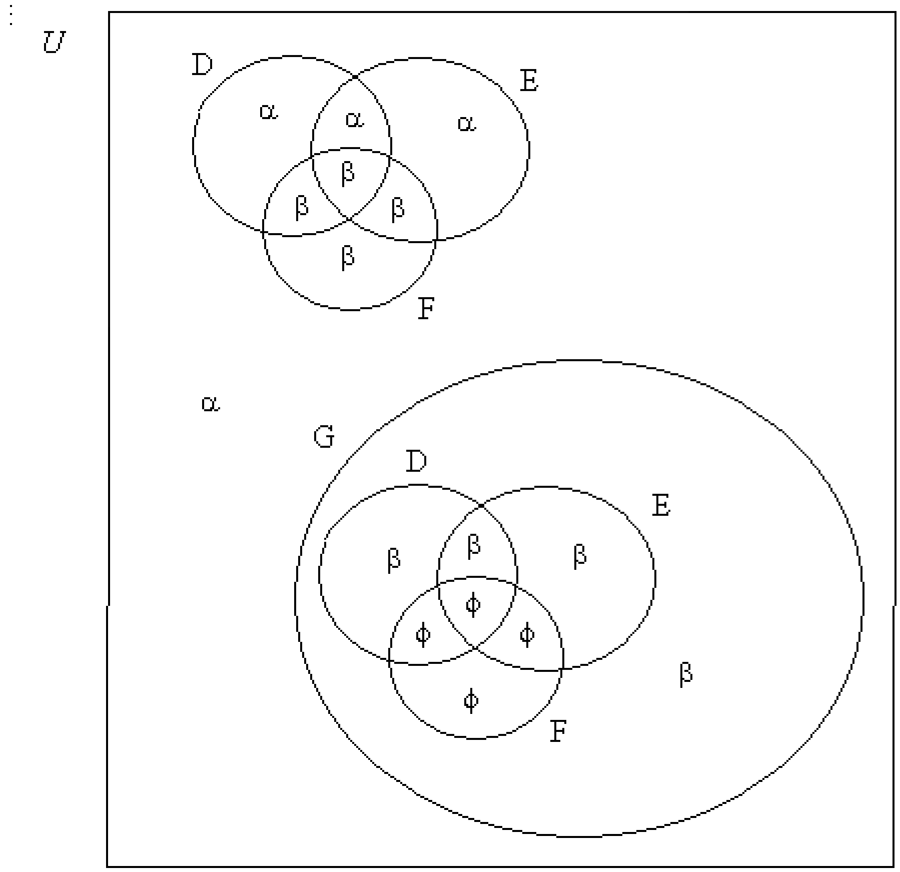
2.
Let the sample space S = U be well defined and let A, B, and C be events.
With the probabilities assigned for events as given in the Venn Diagramme determine if A, B, and C are independent. Create an example for U, A, B, and C suggested by this Venn Diagramme.



3.
Let the sample space S = U be well defined and let D, E, and F be events. With the probabilities assigned for events as given in the Venn Diagramme determine if D, E, and F are independent. Create an example for U, D, E, and F suggested by this Venn Diagramme.



4. Let the sample space S = U be well defined and let D, E, F, and G be events. With the probabilities assigned for events as given in the Venn Diagramme determine if D, E, F, and G are independent. Create an example for U, D, E, F, and G suggested by this Venn Diagramme.



$\alpha = \frac{1}{9}$

$\beta = \frac{1}{18}$

$\phi = \frac{1}{36}$