

Math 302 Probability and Statistics II

Dr. McLoughlin's Class

The Pivotal Method

Handout 3

As we have stated (and understand one hopes) the point to studying probability and statistics is the analysis of data. It is intuitively clear that a point-estimate is useful, but not wholly. So, we consider a method that will yield a confidence interval - - the pivotal method (so named by Dr. J. Walker of Georgia State University to whom I am indebted since I learnt it from him).

Let θ be the parametre of interest we are attempting to estimate. We do not know it (only God knows it) but it is a constant.

First we find an estimator for θ , $\hat{\theta}$, using any method we have previously discussed (the method of moments (MOM), maximum likelihood estimation (MLE), etc.).

Next we find a sampling distribution for $\hat{\theta}$; the distribution of $\hat{\theta}$ should depend on θ (else it be a poor estimator of θ and so not 'reliable.').

Third, we find, create, or produce a function, $g((\hat{\theta}, \theta))$, which involves no other unknown quantities (for example for an estimator $\hat{\alpha}$ for the parametre α in an Erlang distribution, a function $f((\hat{\alpha}, \alpha))$ would not be of use if it was with respect to α and β where both are unknown and we are attempting to estimate α with $\hat{\alpha}$). $g((\hat{\theta}, \theta))$ should involve $\hat{\theta}$ and θ and may involve other sample quantities, other known quantities so that its probabilities for $\hat{\theta}$ (the distribution) can be found by tabular, analytical, or computational methods. The quantity (function), $g((\hat{\theta}, \theta))$, is called the pivot quantity.

Now, we use the known distribution of $G = g((\hat{\theta}, \theta))$ to find two constants, L and U, such that $\Pr(L < G < U) = 1 - \alpha$ or $\Pr(L < G < U) \approx 1 - \alpha$ (for some α value) [in practice this will almost always be approximately (you know why)].

The next procedure is to solve each inequality ($L < G \wedge G < U$) so we get equivalent probability descriptions (functions with respect to the sample) of a lower and upper bound for the $\hat{\theta}$. Let $g_L((X_1, X_2, X_3, \dots, X_n))$ solves the inequality $L < G$ and $g_U((X_1, X_2, X_3, \dots, X_n))$ solves the inequality $G < U$ so we can produce the $\Pr(g_L((X_1, X_2, X_3, \dots, X_n)) < \hat{\theta} < g_U((X_1, X_2, X_3, \dots, X_n))) = 1 - \alpha$ (really $\approx 1 - \alpha$). So, we see that the inequality $g_L((X_1, X_2, \dots, X_n)) < \hat{\theta} < g_U((X_1, X_2, \dots, X_n))$ produces the $(1 - \alpha) \cdot 100\%$ confidence interval for θ , which is:

$$(g_L((X_1, X_2, X_3, \dots, X_n)), g_U((X_1, X_2, X_3, \dots, X_n))).$$

Considering this from the standpoint of exemplars let us, for example, have a sample $X_1, X_2, X_3, \dots, X_n$ that is i.i.d. from a $\mathcal{N}(X, \mu, \sigma)$ distribution where σ^2 is **known** and let us suppose we want to find a 90% confidence interval for μ (the 90% of course comes from the α level which is set *a priori*) and use the pivotal method to find the confidence interval.

1. So, we use the method of moments (MOM), maximum likelihood estimation (MLE), or another apropos method for estimation of the parametre of interest. In this case, let us use

$\hat{\mu} = \bar{X}$ since it is unbiased (which we proved) and we want to use an unbiased estimator.

2. Since $X \sim \mathcal{N}(X, \mu, \sigma)$ it is therefore the case that \bar{X} is distributed normally with mean of $\mu_{\bar{X}} = \mu$ and standard error of the mean of $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$. So, $\bar{X} \sim \mathcal{N}(\bar{X}, \mu_{\bar{X}}, \sigma_{\bar{X}})$ (we do not have to appeal to the central limit theorem).

3. From normal distribution theory, we know that $g((\hat{\mu}, \mu)) = \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ where

$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = Z \sim \mathcal{N}(Z, 0, 1)$. Note that $g((\hat{\mu}, \mu))$ involves $\hat{\mu} = \bar{X}$ (from the sample), μ (the

parametre to be estimated), σ (which is known), and n (which is known (the sample size)).

Thus, $g((\hat{\mu}, \mu))$ satisfies the pivotal method conditions to be a pivotal quantity.

4. Using the standard normal tables (approximations though they are) we know that $\Pr(-1.645 < Z < 1.645) \doteq .9$; so, letting $L = -1.645$ and $U = 1.645$ we have

$$\Pr(L < Z < U) = \Pr(-1.645 < \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} < 1.645) \doteq 1 - .1 = .9$$

$$\Pr(-1.645 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645) \doteq .9$$

Let us break up the inequality into two parts for ease of operation and because it yields a 'reversed' subsequent to that which on assumes from the direction of the inequality.

$$5A. \text{ Solving } -1.645 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \text{ we get } -1.645 \cdot \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu \Rightarrow -1.645 \cdot \frac{\sigma}{\sqrt{n}} + \mu < \bar{X}$$

$$\Rightarrow \mu < \bar{X} + 1.645 \cdot \frac{\sigma}{\sqrt{n}}$$

$$5B. \text{ Solving } \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645 \text{ we get } \bar{X} - \mu < 1.645 \cdot \frac{\sigma}{\sqrt{n}} \Rightarrow \bar{X} < 1.645 \cdot \frac{\sigma}{\sqrt{n}} + \mu$$

$$\Rightarrow \bar{X} - 1.645 \cdot \frac{\sigma}{\sqrt{n}} < \mu$$

$$6. \text{ So, } \Pr(L < Z < U) = \Pr\left(-1.645 < \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} < 1.645\right) = \Pr\left(-1.645 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645\right) =$$

$$\Pr\left(\bar{X} - 1.645 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.645 \cdot \frac{\sigma}{\sqrt{n}}\right) \doteq .9 \text{ which implies that the corresponding 90\%}$$

$$\text{confidence interval is } \left(\bar{X} - 1.645 \cdot \frac{\sigma}{\sqrt{n}}, \bar{X} + 1.645 \cdot \frac{\sigma}{\sqrt{n}}\right).$$

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