MATH 351 ADVANCED CALCULUS (REAL ANALYSIS) I
DR. MCLOUGHLIN MORE OF A DISCUSSION OF SEQUENCES §16 – §18
MORE DEFINITIONS, AXIOMS, LEMMAS, COROLLARIES, OR THEOREMS
HANDOUT 5

Let $U = \mathbb{R}$

Theorem 18.06: Let sequence $f(\mathrm{n}),\ f: \mathbb{N} \longrightarrow \mathbb{R}$ be well defined and bounded. It is the case that there exists a convergent subsequence of the sequence f, $\{f_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$.

Without loss of generality let us name the convergent subsequence $\{f_{\mathrm{n_i}}\}_{\mathrm{i}\in\mathbb{N}}$ meaning that where

$f = \{(1, f(1)), (2, f(2)), (3, f(3)), (4, f(4)), (5, f(5)), \ldots (\mathrm{m}, f(\mathrm{m})), ((\mathrm{m}+1), f(\mathrm{m}+1)) \ldots\}$

there exists an index set $\mathrm{A} = \{\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3, \ldots, \mathrm{n}_{\mathrm{p}}, \ldots\}$ where $\mathrm{n}_1 < \mathrm{n}_2 < \mathrm{n}_3 < \ldots < \mathrm{a}_{\mathrm{p}} < \ldots$ and A is unbounded above

$f_{\mathrm{n_i}} = \{(\mathrm{n}_1, f(\mathrm{n}_1)), (\mathrm{n}_2, f(\mathrm{n}_2)), (\mathrm{n}_3, f(\mathrm{n}_3)), \ldots, (\mathrm{n}_{\mathrm{k}}, f(\mathrm{n}_{\mathrm{k}})), \ldots\}$

$\{f_{\mathrm{n_i}}\}_{\mathrm{i}\in\mathbb{N}}$ is a convergent subsequence of the sequence $f \Rightarrow f$ is a bounded sequence.

Ms. Hutton's Claim: Let sequence $f(\mathrm{n}),\ f: \mathbb{N} \longrightarrow \mathbb{R}$ be well defined. Let us use the notation $\{f_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$. Let $\{f_{\mathrm{n_i}}\}_{\mathrm{i}\in\mathbb{N}}$ be a convergent subsequence of the sequence f,

$f = \{(1, f(1)), (2, f(2)), (3, f(3)), (4, f(4)), (5, f(5)), \ldots (\mathrm{m}, f(\mathrm{m})), ((\mathrm{m}+1), f(\mathrm{m}+1)) \ldots\}$

$f_{\mathrm{n_i}} = \{(\mathrm{n}_1, f(\mathrm{n}_1)), (\mathrm{n}_2, f(\mathrm{n}_2)), (\mathrm{n}_3, f(\mathrm{n}_3)), \ldots, (\mathrm{n}_{\mathrm{k}}, f(\mathrm{n}_{\mathrm{k}})), \ldots\}$

$\{f_{\mathrm{n_i}}\}_{\mathrm{i}\in\mathbb{N}}$ is a convergent subsequence of the sequence $f \Rightarrow f$ is a bounded sequence.

Also recall:

Definition 16.14: The sequence $f(\mathrm{n}),\ f: \mathbb{N} \longrightarrow \mathbb{R}$, is **Cauchy** if $\forall\ \varepsilon > 0\ \exists\ \mathrm{p} \in \mathbb{N}$ such that $|f(\mathrm{n}) - f(\mathrm{m})| < \varepsilon\ \forall\ \mathrm{n} > \mathrm{p} \wedge \mathrm{m} > \mathrm{p}$ where $\mathrm{m} \in \mathbb{N}$ and $\mathrm{n} \in \mathbb{N}$.

Definition 18.05: The sequence $f(\mathrm{n}),\ f: \mathbb{N} \longrightarrow \mathbb{R}$, has the 'shrinking difference' property if $\forall\ \varepsilon > 0\ \exists\ \mathrm{p} \in \mathbb{N}$ such that $|f(\mathrm{n}) - f(\mathrm{n}+1)| < \varepsilon\ \forall\ \mathrm{n} > \mathrm{p}$.

Theorem 18.05: A sequence converges *iff* it is a Cauchy sequence.

Exercise 18.01: Construct a sequence $f(\text{n}),\ f:\mathbb{N} \longrightarrow \mathbb{R},$ that has the 'shrinking difference' property but is NOT a Cauchy sequence.

We discussed that construction of said would necessitate that it either not be monotonic or not be bounded (why?[1]). We discussed that the sequence we construct needs to be non-convergent (divergent) because of theorem 18.05. We 'played' around constructing different sequences (examples of sequences that were monotonic and bounded; sequences that were monotonic and not bounded; sequences that were bounded and not monotonic; and, sequences that were not monotonic and not bounded.

We discussed that construction of said would be difficult to make the sequence oscillating (not that it can't be done but we opined it was not clear how to proceed); so, we hinted try to construct a sequence that has the 'shrinking difference' property but is NOT a Cauchy sequence such that it does not have a limit and it 'blows up' since 'blow up' or blow down' is but a direction and not essentially different properties.

We noted we need the sequence, therefore, to be not bounded since it needs to 'blow up.' Of course we recalled it

[1] Exercise: Answer why this is the case.