MATH 351
ADVANCED CALCULUS (REAL ANALYSIS) I
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MORE OF A DISCUSSION OF SEQUENCES §16 – §18
MORE DEFINITIONS, AXIOMS, LEMMAS, COROLLARIES, OR THEOREMS
HANDOUT 4 $^7/_8$

Let $U = \mathbb{R}$

Theorem 16.06: Let sequence $f\,(\mathrm{n})$, $f : \mathbb{N} \longrightarrow \mathbb{R}$, converge to the number L, the sequence $h\,(\mathrm{n})$, $h : \mathbb{N} \longrightarrow \mathbb{R}$, converge to the number M, and the sequence $g\,(\mathrm{n})$, $g : \mathbb{N} \longrightarrow \mathbb{R}$, converge to the number J. Suppose $f\,(\mathrm{n}) \le g(\mathrm{n}) \le h(\mathrm{n}) \ \forall \ \mathrm{n} \in \mathbb{N}$.
Then it is the case that $\mathrm{L} \le \mathrm{J} \le \mathrm{M}$.

Claim 18.01: Let sequence $f\,(\mathrm{n})$, $f : \mathbb{N} \longrightarrow \mathbb{R}$, converge to the number L and sequence $h\,(\mathrm{n})$ ∋ $h : \mathbb{N} \longrightarrow \mathbb{R}$ converge to the number M, and, consider the sequence $g\,(\mathrm{n})$, $g : \mathbb{N} \longrightarrow \mathbb{R}$ where $f\,(\mathrm{n}) \le g(\mathrm{n}) \le h(\mathrm{n}) \ \forall \ \mathrm{n} \in \mathbb{N}$. Then $\{g(\mathrm{n})\}_{\mathrm{n}=1}^{\infty}$ converges.
We showed this to be false.

Claim 18.02: Let sequence $f\,(\mathrm{n})$, $f : \mathbb{N} \longrightarrow \mathbb{R}$, converge to the number L and sequence $h\,(\mathrm{n})$ ∋ $h : \mathbb{N} \longrightarrow \mathbb{R}$ converge to the number M, and, consider the sequence $g\,(\mathrm{n})$, $g : \mathbb{N} \longrightarrow \mathbb{R}$ where $f\,(\mathrm{n}) \le g(\mathrm{n}) \le h(\mathrm{n}) \ \forall \ \mathrm{n} \in \mathbb{N}$. Then $\{g(\mathrm{n})\}_{\mathrm{n}=1}^{\infty}$ is monotonic.
We showed this to be false.

Claim 18.03: Let sequence $f\,(\mathrm{n})$, $f : \mathbb{N} \longrightarrow \mathbb{R}$, converge to the number L and sequence $h\,(\mathrm{n})$ ∋ $h : \mathbb{N} \longrightarrow \mathbb{R}$ converge to the number M, and, consider the sequence $g\,(\mathrm{n})$, $g : \mathbb{N} \longrightarrow \mathbb{R}$ where $f\,(\mathrm{n}) \le g(\mathrm{n}) \le h(\mathrm{n}) \ \forall \ \mathrm{n} \in \mathbb{N}$. Then $\{g(\mathrm{n})\}_{\mathrm{n}=1}^{\infty}$ is bounded.
We outlined the proof of why this is true.

Claim 18.04: Let sequence $\{f(\mathrm{n})\}_{\mathrm{n}=1}^{\infty} \xrightarrow{\mathrm{n}\longrightarrow\infty} \mathrm{M}$, (f converge to the number M) and sequence $\{h(\mathrm{n})\}_{\mathrm{n}=1}^{\infty} \xrightarrow{\mathrm{n}\longrightarrow\infty}$ and, consider the sequence g (n), $g : \mathbb{N} \longrightarrow \mathbb{R}$ where $f(\mathrm{n}) \leq g(\mathrm{n}) \leq h(\mathrm{n})$ $\forall$ n $\in \mathbb{N}$. Then $\{g(\mathrm{n})\}_{\mathrm{n}=1}^{\infty}$ converges and it converges to M.

We began the outline of the proof of why this is true.

Definition 18.10: Consider the sequence f, $f : \mathbb{N} \longrightarrow \mathbb{R}$. Clearly $dom(f) = \mathbb{N}$

let $\mathrm{A} \subseteq \mathbb{N}$ such that $|\mathrm{A}| = \aleph_0$ and further A is well ordered by the relation $\leq$.

Therefore, we can understand that $\mathrm{A} = \{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \ldots, \mathrm{a}_\mathrm{p}, \ldots\}$ where $\mathrm{a}_1 \leq \mathrm{a}_2 \leq \mathrm{a}_3 \leq \ldots \leq \mathrm{a}_\mathrm{p} \leq \ldots$
but let us moreover insist that $\mathrm{a}_1 < \mathrm{a}_2 < \mathrm{a}_3 < \ldots < \mathrm{a}_\mathrm{p} < \ldots$

then we say that the sequence g, $g : \mathrm{A} \longrightarrow \mathbb{R}$ is a subsequence of the sequence f, $f : \mathbb{N} \longrightarrow \mathbb{R}$.

Understood that then what is the case is that $g = f|_{\mathrm{A}}$

$f = \{(1, f(1)), (2, f(2)), (3, f(3)), \ldots, (\mathrm{m}, f(\mathrm{m})), \ldots\}$

$g = \{(\mathrm{a}_1, f(\mathrm{a}_1)), (\mathrm{a}_2, f(\mathrm{a}_2)), (\mathrm{a}_3, f(\mathrm{a}_3)), \ldots, (\mathrm{a}_\mathrm{m}, f(\mathrm{a}_\mathrm{m})), \ldots\}$

The idea is, for example:

$f =$
$\{(1, f(1)), (2, f(2)), (3, f(3)), (4, f(4)), (5, f(5)), (6, f(6)), (7, f(7)), (8, f(8)), (9, f(9)) \ldots, (\mathrm{m}, f(\mathrm{m})), \ldots\}$

$g = \{(2, f(2)), (3, f(3)), (6, f(6)), (9, f(9)), (27, f(27)), (55, f(55)) \ldots, (\mathrm{m}, f(\mathrm{m})), \ldots\}$