MATH 351
ADVANCED CALCULUS (REAL ANALYSIS) I
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MORE OF A DISCUSSION OF SEQUENCES §16 – §18
MORE DEFINITIONS, AXIOMS, LEMMAS, COROLLARIES, OR THEOREMS
HANDOUT 4 ½

Let $U = \mathbb{R}$

Recall:

Notation: The sequence f, $f:\mathbb{N}\longrightarrow\mathbb{R}$ converges to the number L is written as $f(\text{n})\longrightarrow \text{L}$ or

$f(\text{n})\xrightarrow{\text{n}\to\infty}\text{L}$ or $\lim\limits_{\text{n}\longrightarrow\infty} f(\text{n}) = \text{L}$

If the sequence f, $f:\mathbb{N}\longrightarrow\mathbb{R}$ diverges we write $\lim\limits_{\text{n}\longrightarrow\infty}\left(f(\text{n})\right)$ does not exist.

$\lim\limits_{\text{n}\longrightarrow\infty} f(\text{n}) = -\infty$ or $\lim\limits_{\text{n}\longrightarrow\infty} f(\text{n}) = \infty$ is wrong!!!!!

Definition 16.07: The sequence $f:\mathbb{N}\longrightarrow\mathbb{R}$ is bounded if there exists a $\text{p}\in\mathbb{R}$

such that $|f(\text{n})| < \text{p}\ \forall\ \text{n}\in\mathbb{N}$.

Theorem 16.01: Let sequence $f(\text{n})$, $f:\mathbb{N}\longrightarrow\mathbb{R}$, converge to the number L and let k be a real number. Then $\text{k}(\lim\limits_{\text{n}\longrightarrow\infty} f(\text{n})) = (\lim\limits_{\text{n}\longrightarrow\infty}(\text{k}\cdot f(\text{n}))) = (\text{k}\cdot\text{L}) = \text{k}\cdot\text{L} = \text{kL}$

Theorem 16.02: Let sequence $f(\text{n})$, $f:\mathbb{N}\longrightarrow\mathbb{R}$, converge to the number L and let k be a real number. Then $\text{k} + (\lim\limits_{\text{n}\longrightarrow\infty} f(\text{n})) = (\lim\limits_{\text{n}\longrightarrow\infty}(\text{k} + f(\text{n}))) = (\text{k}+\text{L}) = \text{k}+\text{L}$

Theorem 16.03: Let sequence $f(\text{n})$, $f:\mathbb{N}\longrightarrow\mathbb{R}$, converge to the number L and sequence $g(\text{n})$, $g:\mathbb{N}\longrightarrow\mathbb{R}$, converge to the number M. Then

$$(\lim_{\text{n}\longrightarrow\infty}(f(\text{n}) + g(\text{n})) = (\lim_{\text{n}\longrightarrow\infty}(f(\text{n}))) + (\lim_{\text{n}\longrightarrow\infty}(g(\text{n}))) = (\text{L}+\text{M}) = \text{L}+\text{M}$$

Theorem 16.04: Let sequence $f(\text{n})$, $f:\mathbb{N}\longrightarrow\mathbb{R}$, converge to the number L and sequence $g(\text{n})$, $g:\mathbb{N}\longrightarrow\mathbb{R}$, converge to the number M. Then

$$(\lim_{\text{n}\longrightarrow\infty}(f(\text{n})\cdot g(\text{n})) = (\lim_{\text{n}\longrightarrow\infty}(f(\text{n})))\cdot(\lim_{\text{n}\longrightarrow\infty}(g(\text{n}))) = (\text{L}\cdot\text{M}) = \text{LM}$$

Theorem 16.05: Let sequence $f(\mathrm{n}),\ f:\mathbb{N}\longrightarrow\mathbb{R},$ converge to the number L and sequence $g(\mathrm{n}),\ g:\mathbb{N}\longrightarrow\mathbb{R},$ converge to the number M, $\mathrm{M}\neq 0$; and, $g(\mathrm{n})\neq 0\ \forall\ \mathrm{n}\in\mathbb{N}$. Then

$$(\lim_{\mathrm{n}\longrightarrow\infty}(f(\mathrm{n})\div g(\mathrm{n})) = (\lim_{\mathrm{n}\longrightarrow\infty}(f(\mathrm{n})))\div(\lim_{\mathrm{n}\longrightarrow\infty}(g(\mathrm{n}))) = (\mathrm{L}\div\mathrm{M}) = \frac{\mathrm{L}}{\mathrm{M}}$$

Theorem 16.06: Let sequence $f(\mathrm{n}),\ f:\mathbb{N}\longrightarrow\mathbb{R},$ converge to the number L and sequence $g(\mathrm{n}),\ g:\mathbb{N}\longrightarrow\mathbb{R},$ converge to the number M, and, the sequence $h(\mathrm{n}),\ h:\mathbb{N}\longrightarrow\mathbb{R},$ converge to the number J. Suppose $f(\mathrm{n})\leq g(\mathrm{n})\leq h(\mathrm{n})\ \forall\ \mathrm{n}\in\mathbb{N}$. Then $\mathrm{L}\ \leq\ \mathrm{M}\ \leq\ \mathrm{J}$.

Theorem 16.07: Let sequence $f(\mathrm{n}),\ f:\mathbb{N}\longrightarrow\mathbb{R},$ be well defined such that $f(\mathrm{n}) > 0\ \forall\ \mathrm{n}$. Let the sequence $g(\mathrm{n})$ be defined as $g(\mathrm{n}) = \dfrac{f(\mathrm{n}+1)}{f(\mathrm{n})}$, and let $g:\mathbb{N}\longrightarrow\mathbb{R},$ converge to the number M, where $\mathrm{M} < 1$. Then $\lim_{\mathrm{n}\longrightarrow\infty} f(\mathrm{n}) = 0$.

Definition 16.08: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\rightarrow\mathbb{R},$ is **increasing** if $f(\mathrm{n}) < f(\mathrm{n}+1)\ \forall\ \mathrm{n}\in\mathbb{N}$.

Definition 16.09: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\rightarrow\mathbb{R},$ is **non-decreasing** if $f(\mathrm{n})\leq f(\mathrm{n}+1)\ \forall\ \mathrm{n}\in\mathbb{N}$.

Definition 16.10: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\rightarrow\mathbb{R},$ is **decreasing** if $f(\mathrm{n}) > f(\mathrm{n}+1)\ \forall\ \mathrm{n}\in\mathbb{N}$.

.Definition 16.11: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\rightarrow\mathbb{R},$ is **non-increasing** if $f(\mathrm{n})\geq f(\mathrm{n}+1)\ \forall\ \mathrm{n}\in\mathbb{N}$.

.Definition 16.12: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\rightarrow\mathbb{R},$ is **constant** if if $f(\mathrm{n}) = f(\mathrm{n}+1)\ \forall\ \mathrm{n}\in\mathbb{N}$.

Definition 16.13: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\longrightarrow\mathbb{R},$ is **monotonic** (or monotone) if it is increasing or non-decreasing.[1]

Theorem 16.08: A monotonic sequence converges *iff* it is bounded.

Definition 16.14: The sequence $f(\mathrm{n}),\ f:\mathbb{N}\longrightarrow\mathbb{R},$ is **Cauchy** if

$\forall\ \varepsilon > 0\ \exists\ \mathrm{p}\in\mathbb{N}$ such that $|f(\mathrm{n}) - f(\mathrm{m})| < \varepsilon\ \forall\ \mathrm{n} > \mathrm{p}\wedge\ \mathrm{m} > \mathrm{p}$.

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[1] Note that non-increasing, increasing, decreasing, constant, and non-decreasing are contained within this definition so we don't have to expand it to mention all 5 (they are 'covered,' eh?).

Nomenclature: A sequence $f(n),\ f:\mathbb{N} \longrightarrow \mathbb{R}$ that is Cauchy is called a Cauchy Sequence (isn't that astonishing?).

Some Corrections to the Text:

Definition 18.1: The sequence $\{s_n\}$ is increasing if $s_n \le s_{(n+1)} \forall n \in \mathbb{N}$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is decreasing if $s_n \ge s_{(n+1)} \forall n \in \mathbb{N}$ is wrong!!!!!!

Definition 18.1A: The sequence $\{s_n\}_{n=1}^{\infty}$ is increasing if $s_n < s_{(n+1)} \forall n \in \mathbb{N}$.

Definition 18.1B: The sequence $\{s_n\}_{n=1}^{\infty}$ is non-decreasing if $s_n \le s_{(n+1)} \forall n \in \mathbb{N}$.

Definition 18.1C: The sequence $\{s_n\}_{n=1}^{\infty}$ is non-increasing if $s_n \ge s_{(n+1)} \forall n \in \mathbb{N}$.

Definition 18.1D: The sequence $\{s_n\}_{n=1}^{\infty}$ is decreasing if $s_n > s_{(n+1)} \forall n \in \mathbb{N}$.

Exercise 18.15 insert the word monotonic after every and before contractive (line 3)