MATH 351
ADVANCED CALCULUS (REAL ANALYSIS) I
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BEGINNING DISCUSSION OF SEQUENCES §16
HANDOUT IV

Let $U = \mathbb{R}$

Review your notes and work on sequences from Math 224. Recall some important concepts that create our understanding of sequences:

Definition 9 (Handout 1): Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The real number L is the limit of the function $f: A \longrightarrow B$ at $a \in A$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < |x - a| < \delta$.

Definition 16 - 00 : A function $f: A \longrightarrow B$ (from A to B) is a relation such that $\text{dom}(f) = A$ and $(a_1, b_1) \in f \wedge (a_1, b_2) \in f \Rightarrow b_1 = b_2$.

Definition 16.01: A sequence, f_n or $f(n)$, is a function $f: A \longrightarrow \mathbb{R}$. $\text{dom}(f) \subseteq \mathbb{N}$ $\text{ran}(f) \subset \mathbb{R}$

Definition 16.02: A finite sequence, f_n or $f(n)$, is a function $f: A \longrightarrow \mathbb{R}$. $\text{dom}(f) \subseteq \mathbb{N}$,

$\text{ran}(f) \subset \mathbb{R} \wedge |\text{dom}(f)| = |A| < \aleph_0$.

As we said, a finite sequence is such that $|\text{dom}(f)| < \aleph_0$. They are uninteresting. Further, $\exists$ a bijection, g: $\text{dom}(f) \longleftrightarrow \mathbb{N}_p$ for some $p \in \mathbb{N}$ So, we have (without loss of generality):

Alternate Definition 16.02: A finite sequence, f_n or $f(n)$, is a function $f: \mathbb{N}_p \longrightarrow \mathbb{R}$ for some $p \in \mathbb{N}$

Definition 16.03: An infinite sequence, f_n or $f(n)$, is a function $f: A \longrightarrow \mathbb{R}$. $\text{dom}(f) \subseteq \mathbb{N}$

$\text{ran}(f) \subset \mathbb{R} \wedge |\text{dom}(f)| = |A| = \aleph_0$.

Further, $\exists$ a bijection, h: $\text{dom}(f) \longleftrightarrow \mathbb{N}$

So, we have (without loss of generality):

Alternate Definition 16.03: An infinite sequence, f_n or $f(n)$, is a function $f: \mathbb{N} \longrightarrow \mathbb{R}$. $\text{cod}(f) \subset \mathbb{R}$

The author of the text we use (Lay) refers to an infinite sequence as a sequence since the finite case is uninteresting, we shall adopt that nomenclature. Why are finite sequences uninteresting? Because:

Definition 16.04: The number L is the **limit of the sequence** $f : \mathbb{N} \longrightarrow \mathbb{R}$. iff for every $\varepsilon > 0$ there exists a $m \in \mathbb{N}$ such that $|f(n) - L| < \varepsilon$ whenever $n > m$.

Alternate Definition 16.04: The number L is the **limit of the sequence** $f : \mathbb{N} \longrightarrow \mathbb{R}$. iff for every segment S containing the number L there exists a $m \in \mathbb{N}$ such that $f(n) \in S$ whenever $n > m$.

Definition 16.05: The sequence f, $f : \mathbb{N} \longrightarrow \mathbb{R}$, is said to **converge** to the number L iff for every $\varepsilon > 0$ there exists a $m \in \mathbb{N}$ such that $|f(n) - L| < \varepsilon$ whenever $n > m$.

Notation: The sequence f, $f : \mathbb{N} \longrightarrow \mathbb{R}$, converges to the number L is written as $f(n) \longrightarrow L$ or $f(n) \xrightarrow{n \to \infty} L$ or $\lim_{n \longrightarrow \infty} f(n) = L$

Definition 16.06: The sequence $f : \mathbb{N} \longrightarrow \mathbb{R}$. is said to **diverge** iff it is not the case that there exists a number L such that for every $\varepsilon > 0$ there exists a $m \in \mathbb{N}$ such that $|f(n) - L| < \varepsilon$ whenever $n > m$.

If the sequence f, $f : \mathbb{N} \to \mathbb{R}$, diverges we write $\lim_{n \longrightarrow \infty} f(n)$ does not exist.

$\lim_{n \longrightarrow \infty} f(n) = -\infty$ or $\lim_{n \longrightarrow \infty} f(n) = \infty$ is wrong!!!!!

Definition 16.07: The sequence $f : \mathbb{N} \longrightarrow \mathbb{R}$ is **bounded** if there exists a $p \in \mathbb{R}$ such that $|f(n)| < p \; \forall \; n \in \mathbb{N}$.