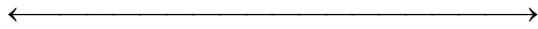


**MATH 351 HANDOUT 1**  
**THE AXIOMS OF THE REALS**  
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Recall from Set Theory that every set,  $A$ , is a subset of some well defined universe,  $U$ .

The universe we will be discussing at the onset of Math 351 is the real line.



There is *no* centre (e.g.: the nonsense about  $\infty + (-\infty) = 0$  one may have learnt in high school is a fallacy)

The reals (as all of math) begins with axioms (that you **must** memorise):

**The Field Axioms of  $\mathbb{R}$**

Axiom 1 (closure of addition):  $\forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$  and  $(x = w \wedge y = v) \Rightarrow (x + y = w + v)$

Axiom 2 (commutative of addition):  $\forall x, y \in \mathbb{R}, x + y = y + x$ .

Axiom 3 (associative of addition):  $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$

Axiom 4 (existence of identity of addition):  $\exists$  a unique number  $0 \ni x + 0 = x \quad \forall x \in \mathbb{R}$

Axiom 5 (existence of additive inverse):  $\forall x \in \mathbb{R} \exists$  a unique number  $-x \ni x + (-x) = 0$

Axiom 6 (closure of multiplication):  $\forall x, y \in \mathbb{R}, x \cdot y \in \mathbb{R}$  and  $(x = w \wedge y = v) \Rightarrow (x \cdot y = w \cdot v)$

Axiom 7 (commutative of multiplication):  $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$ .

Axiom 8 (associative of multiplication):  $\forall x, y, z \in \mathbb{R}, (x \cdot y) \cdot z = x \cdot (y \cdot z)$

Axiom 9 (existence of identity of multiplication):  $\exists$  a unique number  $1 \ni x \cdot 1 = x \quad \forall x \in \mathbb{R}$

( $1 \neq 0$ ).

Axiom 10 (existence of multiplicative inverse):  $\forall x \in \mathbb{R} \ni x \neq 0 \exists$  a unique number  $x^{-1}$

$\ni x \cdot (x^{-1}) = 1$

Axiom 11 (distributive of multiplication over addition):  $\forall x, y, z \in \mathbb{R}, x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

## The Order Axioms of $\mathbb{R}$

Axiom 12 (trichotomy):  $\forall x, y \in \mathbb{R}$ , exactly one of the following relationships exists between  $x$  and  $y$ :  $x < y$ ,  $x = y$ ,  $\vee x > y$ .  $[(x < y) \text{ exor } (x = y) \text{ exor } (x > y)]$

Axiom 13 (transitive):  $\forall x, y, z \in \mathbb{R}$ ,  $[(x < y) \wedge (y < z)] \Rightarrow (x < z)$

Axiom 14 (preservation of order under addition):  $\forall x, y, z \in \mathbb{R}$ ,  $(x < y) \Rightarrow (x + z < y + z)$

Axiom 15 (preservation of order for positive multiplier):  $\forall x, y \in \mathbb{R}$ ,  $[(x < y) \wedge (0 < z)] \Rightarrow (x \cdot z < y \cdot z)$

## The Completeness Axiom of $\mathbb{R}$

Axiom 16 (completeness):  $\forall A \subseteq \mathbb{R} \ni A$  is bounded above  $\exists$  a number,  $m$ , which is the supremum of the set  $A$ .

## Some Definitions One Should Recall That Are Important From Day 1

Let  $D \neq \emptyset \ni D \subseteq \mathbb{R}$ ,  $A \neq \emptyset \ni A \subseteq \mathbb{R}$ ,  $B \neq \emptyset \ni B \subseteq \mathbb{R}$

Definition 1: A function  $f: D \longrightarrow \mathbb{R}$  where  $D \subseteq \mathbb{R}$  is well defined iff

$$1. \text{ dom}(f) = D \quad \text{and} \quad 2. (x, y) \in f \wedge (x, z) \in f \quad \Rightarrow \quad z = y$$

Definition 2: A function  $f$  is non-decreasing iff for every  $x$  and for every  $p$ ,  $x \leq p \Rightarrow f(x) \leq f(p)$

Definition 3: A function  $f$  is increasing iff for every  $x$  and for every  $y$ ,  $x < p \Rightarrow f(x) < f(p)$

Definition 4: A function  $f$  is non-increasing iff for every  $x$  and for every  $p$ ,  $x \leq p \Rightarrow f(x) \geq f(p)$

Definition 5: A function  $f$  is decreasing iff for every  $x$  and for every  $y$ ,  $x < p \Rightarrow f(x) > f(p)$

Definition 6: A function  $f: D \longrightarrow B$  is injective iff for every  $x$  and for every  $p$  in  $D$ ,  $f(x) = f(p) \Rightarrow x = p$

Definition 7: A function  $f: D \longrightarrow B$  is surjective iff for every  $y$  in  $B$  there exists an  $x$  in  $D$  such that  $f(x) = y$

Definition 8: The number  $L$  is the limit of the function  $f: A \rightarrow B$  at  $a \in A$  iff for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - L| < \varepsilon$  whenever  $x \in A$  and  $|x - a| < \delta$ .

Definition 9: A function  $f: A \rightarrow B$  is continuous at  $a \in A$  iff for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - f(a)| < \varepsilon$  whenever  $0 < |x - a| < \delta$ .

Definition 10: A function  $f: A \rightarrow B$  is uniformly continuous on the set  $C \subseteq A$  at  $a \in A$  iff for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  whenever  $x \in C$ ,  $y \in C$ , and  $0 < |x - y| < \delta$ .