

HANDOUT ¾  
THE PEANO AXIOMS  
MATH 351  
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The properties of addition of natural numbers can be derived from a short set of axioms. The axioms are called **the Peano Axioms**:

There exists a set,  $P$ , which is defined by the following four axioms.

Axiom 1: There exists a natural number, call it 1, that is not the successor of any other natural number.

Axiom 2: Every natural number has a unique successor. If  $k \in P$ , then let  $k'$  denote the successor of  $k$ .

Axiom 3: Every natural number except one is the successor of exactly one natural number.

Axiom 4: If  $M$  is a set of natural numbers such that  
(i)  $1 \in M$  and  
(ii) for each  $k \in P$ , if  $k \in M$ , then  $k' \in M$ ,  
then  $P = M$ .

$P$ , of course is  $\mathbb{N}$ .

So, the Peano axioms assert the uniqueness of the naturals that this successor property along with the element 1 creates the entirety of the natural numbers. No matter how you name the set (you can call it Ray, or you can call it Jay, . . .) if it has these properties then it really is the naturals.

From these axioms arise the natural numbers by defining what addition by one means.

Definition 3.6.1: For every  $k \in \mathbb{N}$ , define  $k + 1 = k'$ .

Then, note inductively, the entire understanding of addition flows from this definition (likewise multiplication, etc.).

### THE ARCHIMEDEAN PRINCIPLE OF $\mathbb{N}$

$\mathbb{N}$  is unbounded above; in other words, there does not exist a greatest natural number.

It is worth noting that the Peano axioms and the Archimedean principle provide a justification for the concept of mathematical induction.