Handout $1\frac{1}{2}$

The Lemmas We Need for Basic Proofs

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Let our universe be $\mathbb{R}$.
You may assume:

Lemma 1: $0 < 1$.

Lemma 2: Let $x \in \mathbb{R}$ It is the case that $x \cdot 0 = 0$.

Lemma 3: $(-1) \cdot (-1) = 1$.

Lemma 4: Let $x \in \mathbb{R}$. It is the case that $(-1) \cdot x = -x$

Lemma 5: Let $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is the case that $x - y = x + (-y) = x + -y$.
Definition 1: Let $x \in \mathbb{R}$. It is the case that $x \cdot x = x^2$.

Definition 2: Let $x \in \mathbb{R} \wedge n \in \mathbb{N}$. It is the case that $x \cdot x = x^2$ whilst $\underbrace{x \cdot \ldots \cdot x}_{n\ times} = x^n$.

Law of Exponents: Let $x \in \mathbb{R}$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$.

(1) $x^a \cdot x^b = x^{(a+b)}$

(2) $x^a \cdot b^a = (x \cdot b)^a$

(3) $x^a \div x^b = x^{(a-b)}$ when $x^b \neq 0$

(4) $(x^a)^b = x^{a \cdot b}$

Definition S.1: Let $x \in \mathbb{R}$. It is the case that x is positive if and only if $x > 0$.
Definition S.2: Let $x \in \mathbb{R}$. It is the case that x is non-negative if and only if $x \geq 0$.
Definition S.3: Let $x \in \mathbb{R}$. It is the case that x is non-positive if and only if $x \leq 0$.
Definition S.4: Let $x \in \mathbb{R}$. It is the case that x is negative if and only if $x < 0$.

For Number Theoretic Claims

Properties of Natural, Integers, or Rational Numbers that you may assume:

Closure of addition in $\mathbb{N}$: Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $(m + n) \in \mathbb{N}$.
Closure of multiplication in $\mathbb{N}$: Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $(m \cdot n) \in \mathbb{N}$.

Closure of addition in $\mathbb{Z}$: Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a + b) \in \mathbb{Z}$.
Closure of subtraction in $\mathbb{Z}$: Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(a - b) \in \mathbb{Z}$.
Closure of multiplication in $\mathbb{Z}$: Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $(m \cdot n) \in \mathbb{Z}$.

Closure of addition in $\mathbb{Q}$: Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p + q) \in \mathbb{Q}$.
Closure of subtraction in $\mathbb{Q}$: Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p - q) \in \mathbb{Q}$.
Closure of multiplication in $\mathbb{Q}$: Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $(p \cdot q) \in \mathbb{Q}$.
Closure of non-zero division in $\mathbb{Q}$: Let $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ where $q \neq 0$, then $\frac{p}{q} \in \mathbb{Q}$.

Odd or Even Natural Numbers

Definition 3: Let $m \in \mathbb{N}$. m is even if and only if it is the case that there is some natural number j (meaning $j \in \mathbb{N}$) such that $m = 2 \cdot j$.

Definition 4: Let $m \in \mathbb{N}$. m is odd if and only if it is the case that there is some natural number j (meaning $j \in \mathbb{N}$) such that $m = 2 \cdot j - 1$.

Odd or Even Integers

Definition 5: Let $w \in \mathbb{Z}$. w is even if and only if it is the case that there is some integer p (meaning $p \in \mathbb{Z}$) such that $w = 2 \cdot p$.

Definition 6: Let $w \in \mathbb{Z}$. w is odd if and only if it is the case that there is some integer p (meaning $p \in \mathbb{Z}$) such that $w = 2 \cdot p + 1$.

Definition 6 version 2: Let $w \in \mathbb{Z}$. w is odd if and only if it is the case that there is some integer q (meaning $q \in \mathbb{Z}$) such that $w = 2 \cdot q - 1$.

You may not assume (start trying to prove):

Lemma 6: Let $x \in \mathbb{R} \ni x \geq 0$, $y \in \mathbb{R} \ni y \geq 0$.
It is the case that $\sqrt{x}$ exists, $\sqrt{y}$ exists, $\wedge$ $\sqrt{x} \cdot \sqrt{y} = \sqrt{x \cdot y}$.

Lemma 7: Let $x \in \mathbb{R} \ni x \geq 0$. It is the case that $\sqrt{x} \geq 0$.

Lemma 8: Let $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is the case that $x = 0 \wedge y = 0 \implies x = 0 \vee y = 0$.

Lemma 9: Let $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is the case that $x \cdot y = 0 \implies x = 0 \wedge y = 0$.

Lemma 10: Let $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is the case that $x \cdot y \neq 0 \implies x \neq 0 \vee y \neq 0$.

If there are **any other** seemingly 'obvious' definitions, lemmas, theorems, corollaries, laws, etc. *you wish to cite for a proof for class, please ask about it as soon as possible.*