ADVANCED CALCULUS (REAL ANALYSIS) I
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MORE DEFINITIONS, AXIOMS, AND THEOREMS § 16 – §18
HANDOUT 4 ¾

Let $U = \mathbb{R}$.

Recall we have (without loss of generality):

Definition 16.03: An infinite sequence, f_n or $f(n)$, is a function $f : \mathbb{N} \longrightarrow \mathbb{R}$. $\text{ran}(f) \subset \mathbb{R}$

We need distinguish between and betwixt the sequence and the range of the function since the sequence is ordered and the range obviously is not.

Example: Let $\{s_n\}_{n=1}^{\infty}$ be the well defined sequence such that $s_n = (-1)^n$.

$\text{ran}(s) = \{-1, 1\}$, $\text{cor}(s) = \mathbb{N}$, $\text{dom}(s) = \mathbb{N}$, $\text{cod}(s) = \mathbb{R}$

But the sequence is really $\{(1, -1), (2, 1), (3, -1), (4, 1), \ldots\}$ or alternately (and sloppily) $\{-1, 1, -1, 1, -1, 1, \ldots\}$

Recall also:

Definition 16.13: The sequence $f(n)$, $f : \mathbb{N} \longrightarrow \mathbb{R}$, is **monotonic** (or monotone) if it is non-increasing or non-decreasing.[1]

Theorem 16.08: A monotonic sequence converges *iff* it is bounded.

Definition 16.14: The sequence $f(n)$, $f : \mathbb{N} \longrightarrow \mathbb{R}$, is a **Cauchy sequence** if

$\forall\ \varepsilon > 0\ \exists\ p \in \mathbb{N}$ such that $|f(n) - f(m)| < \varepsilon\ \forall\ n > p \wedge m > p$.

Definition 18.01A: The sequence $\{s_n\}_{n=1}^{\infty}$ is increasing if $s_n < s_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Definition 18.01B: The sequence $\{s_n\}_{n=1}^{\infty}$ is non-decreasing if $s_n \leq s_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Definition 18.01C: The sequence $\{s_n\}_{n=1}^{\infty}$ is non-increasing if $s_n \geq s_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Definition 18.01D: The sequence $\{s_n\}_{n=1}^{\infty}$ is decreasing if $s_n > s_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Definition 18.01E: The sequence $\{s_n\}_{n=1}^{\infty}$ is constant if $s_n = s_{(n+1)}\ \forall\ n \in \mathbb{N}$.

Theorem 18.01: The sequence $\{s_n\}_{n=1}^{\infty}$ is constant *iff* it is both non-increasing and non-decreasing.

Theorem 18.02: Let $\{s_n\}_{n=1}^{\infty}$ be a constant sequence. It converges.

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[1] Note that non-increasing, increasing, decreasing, constant, and non-decreasing are contained within this definition so we don't have to expand it to mention all 5 (they are 'covered,' eh?).

Theorem 18.03A: The increasing sequence $\{s_n\}_{n=1}^{\infty}$ is non-decreasing.

Theorem 18.03B: The decreasing sequence $\{s_n\}_{n=1}^{\infty}$ is non-increasing.

Definition 18.04: The sequence $f(n),\ f:\mathbb{N} \longrightarrow \mathbb{R},$ is **eventually monotonic** (or eventually monotone) if $\exists\ p \in \mathbb{N}$ such that it is non-increasing or non-decreasing $\forall\ n > p.$

Theorem 18.04: A sequence converges if it is bounded and eventually monotonic.

Theorem 18.05: A sequence converges *iff* it is a Cauchy sequence.

Definition 18.05: The sequence $f(n),\ f:\mathbb{N} \longrightarrow \mathbb{R},$ has the 'shrinking difference' property if $\forall\ \varepsilon > 0\ \exists\ p \in \mathbb{N}$ such that $|\, f(n) - f(n+1)\,| < \varepsilon\ \forall\ n > p.$

Exercise 18.01: Construct a sequence $f(n),\ f:\mathbb{N} \longrightarrow \mathbb{R},$ that has the 'shrinking difference' property but is NOT a Cauchy sequence.