HANDOUT V

More On The Topology of $\mathbb{R}$ and More Definitions, Lemmas, Theorems, and Corollaries

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Definition: A point x is a **limit point** of a set S if $\forall \varepsilon > 0$, $N(x, \varepsilon) \cap S - \{x\} \neq \emptyset$. The set of all limit points of S is denoted S'.

Definition: Let $S \subseteq \mathbb{R}$. Then the **closure** of S, denoted $\overline{S}$, is defined by $\overline{S} = S \cup S'$.

Definition: A set S is **closed** iff $S = \overline{S}$.

Definition: A set S is **bounded** if is bounded above and bounded below.

Theorem: (Archimedean Property of $\mathbb{R}$) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Definition: A set $S \subseteq T$ is **dense** in T if $\overline{S} = T$.

Definition (Alternate): A set $S \subseteq T$ is **dense** in T iff $\forall x \in T$ and $\forall \varepsilon > 0$, $\exists z \in S \ni$ $z \neq x \quad \wedge \quad d(x, z) < \varepsilon$.

Definition (Second Alternate): A set $S \subseteq T$ is **dense** in T iff $\forall x \in T$ and $\forall$ segments, G, which contain x $\exists z \in S \ni z \in G \wedge z \neq x$.

Definition: A point $x \in S$ and $x \notin S'$ is called an **isolated point**.

Definition: A set S is **perfect** iff A is closed and A contains no isolated points.

Definition: A set S is **disconnected** iff there exists an open set U and an open set V such that $S \subseteq U \cup V$, $U \cap V = \emptyset$, $S \cap U \neq \emptyset$, and $S \cap V \neq \emptyset$.

Definition: A set S that is not disconnected is **connected**.

Definition: A set S is **totally disconnected** iff for every $x \in S$ and $y \in S$ such that $x \neq y$ there exists an open set V such that $U \cap V = \emptyset$, $x \in U$, $y \in V$, and $S \subseteq U \cup V$.

Definition: A set S is **nowhere dense** in $\mathbb{R}$ iff $\overline{S^C}$ is dense in $\mathbb{R}$.

Definition (Alternate): A set $S \subseteq T$ is **nowhere dense** in T iff $\overline{S}$ contains no segments.

Definition: Let $S \subseteq \mathbb{R}$. A set S is **compact** iff every open cover of that set contains a finite subcover.

Theorem: (H - B) A subset S of $\mathbb{R}$ is **compact** iff S is closed and bounded.

Definition: Let $S \subseteq \mathbb{R}$. A set S is **a continuum** iff it is compact and connected.

Theorem: (B - W) Let S be a bounded subset of $\mathbb{R}$ such that $|S| \geq \aleph_0$. It is the case that $\exists p \in \mathbb{R} \ni p \in S'$.