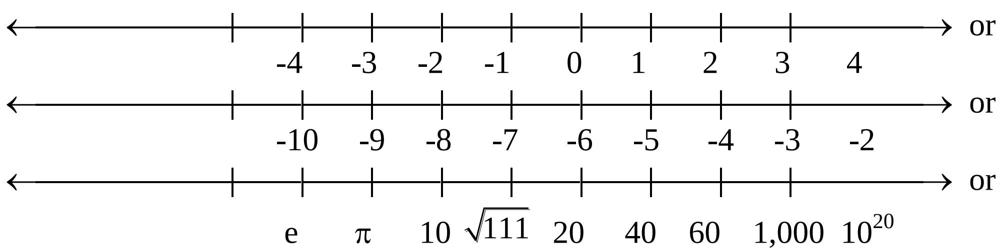
HANDOUT 3
MATH 351 REAL ANALYSIS I
WORKING COPY
DR. MCLOUGHLIN'S CLASS
ON THE TOPOLOGY OF $\mathbb{R}$

Now, let us investigate the wondrous world of $\mathbb{R}$:

There is ***no*** centre (e.g.: the nonsense about $\infty + (-\infty) = 0$ one may have learnt in high school is a fallacy) so one can reasonably represent the line as:



realise that the line is infinite but that if $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then the distance between x and y is finite. So, $|x - y| = \lambda$ where $\lambda \in \mathbb{R}$.

We will adopt the standard of using Greek lower case letters to represent distance and English lower case letters to represent points on the line.

Recall, from what we have studied so far there are some basic sets in $\mathbb{R}$ of particular note:

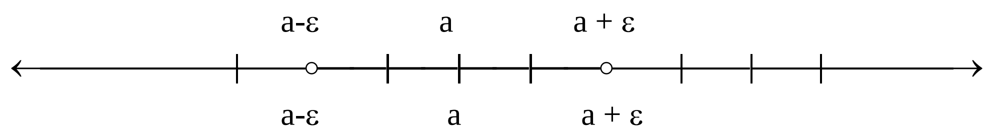
A segment. $a \in \mathbb{R} \wedge b \in \mathbb{R} \ni a < b$ is $(a, b) = \{x \mid a < x < b\}$

An interval $a \in \mathbb{R} \wedge b \in \mathbb{R} \ni a \le b$ is $[a, b] = \{x \mid a \le x \le b\}$

A singleton set. $a \in \mathbb{R}$ is $\{a\} = [a, a]$.

The author of the text, Lay, likes to talk about neighbourhoods and deleted neighbourhoods.

Consider ℝ. Let $a \in \mathbb{R}$. An ε-neighbourhood about a is a really just a segment with radius ε centred at a. So, the neighbourhood (a - ε, a +ε) is a segment with diameter (length) 2ε whose centre is a. Lay denotes the neighbourhood (a - ε, a +ε) as *N* (a, ε).

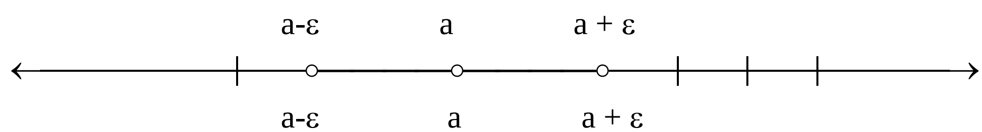


So, $N(a, \varepsilon) = \{x : |x - a| < \varepsilon\}$.

Then Lay talks about deleted neighbourhoods (which is really a waste).

Consider ℝ. Let $a \in \mathbb{R}$. An ε-deleted neighbourhood about a is a really just a two segments segment with radius ε/2 centred at a. So, the neighbourhood (a - ε, a) is one of them and the segment (a, a + ε) is the other segment. Each has diameter (length) ε.

Lay denotes the deleted neighbourhood $(a - \varepsilon, a) \cup (a, a + \varepsilon)$ as *N* *(a, ε). Notice that a is deleted from *N* (a, ε) to create *N* *(a, ε).



So, $N^*(a, \varepsilon) = \{x : |x - a| < \varepsilon \text{ where } x \neq a\} = \{x : 0 < |x - a| < \varepsilon\}$.

What we should notice is that we can define all other terms using just neighbourhoods and we don't need deleted neighbourhoods.

<u>Definition 14.01</u>: Consider $A \subseteq \mathbb{R}$. Let $x \in \mathbb{R}$. x is a **limit point of the set A** (also called an accumulation point of A) if $\forall\, \varepsilon > 0$ the neighbourhood (x - ε, x +ε), N(x, ε), has the property that it contains at least one point of A *distinct from x*.[1]

<u>Definition 14.01.01</u>: The set of accumulation points of S is denoted by S′ and is called the **derived set of S**.

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[1] x may or may not be an element of A - - that isn't a part of the definition of limit point.

Definition 14.02: Consider $A \subseteq \mathbb{R}$. Let $a \in A$. a is an **interior point** of the set A if $\exists$ a segment about a which is a subset of A. So, this means $\exists$ an $\varepsilon > 0 \ni$ the neighbourhood $(a - \varepsilon, a + \varepsilon) \subseteq A$.

Definition 14.03: Consider $A \subseteq \mathbb{R}$. The set of all interior points of the set A is designated int(A).

Definition 14.04: Consider $A \subseteq \mathbb{R}$. Let $a \in A$. a is a boundary point of the set A if $\forall\, \varepsilon > 0$ the segment $(a - \varepsilon, a + \varepsilon)$, (the neighbourhood) $N(a, \varepsilon)$, has the property that

$N(a, \varepsilon) \cap (A - \{a\}) \neq \varnothing \wedge N(a, \varepsilon) \cap ((\mathbb{R} \setminus A) - \{a\}) \neq \varnothing$.

$(a - \varepsilon, a + \varepsilon) \cap (A - \{a\}) \neq \varnothing \wedge (a - \varepsilon, a + \varepsilon) \cap ((\mathbb{R} \setminus A) - \{a\}) \neq \varnothing$.

This means that every segment containing the point a contains a point of A distinct from the point a and a point of A^C distinct from the point a.

Definition 14.05: Consider $A \subseteq \mathbb{R}$. The set of all boundary points of the set A is designated bd(A).

Definition 14.06: Consider $A \subseteq \mathbb{R}$. A is called a **closed** set iff $bd(A) \subseteq A$

Definition 14.07: Consider $A \subseteq \mathbb{R}$. A is called an **open** set iff $bd(A) \subseteq \mathbb{R} \setminus A$

Definition 14.08: The **closure** of the set S is $S \cup S'$ and is denoted as cl(S).

Definition 14.07A: A subset S of $\mathbb{R}$ is **open** in $\mathbb{R}$ iff $\forall\, x \in S, \exists\, \varepsilon > 0 \ni$

the segment $(x - \varepsilon, x + \varepsilon)$, $\subseteq S$.

Theorem 14.01: The union of any arbitrary collection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

Theorem 14.02: The intersection of any finite collection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

<u>Theorem 14.03</u>: Consider $B \subseteq \mathbb{R}$, the point $x \in \mathbb{R}$ is a limit point of B iff any of the following conditions are met:

Every neighborhood of x contains a point distinct from x that is contained in B.

$\exists$ a sequence of distinct points of B that converges to x.

Every neighborhood of x contains infinitely many points of B.

<u>Theorem 14.04</u>: Consider $B \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}$:

$cl(cl(B)) = cl(B)$

$cl(D \cup B) = cl(D) \cup cl(B)$

$cl(B \cap D) \subseteq cl(B) \cap cl(D)$

<u>Theorem 14.05</u>: A subset A of $\mathbb{R}$ is closed if any of the following conditions are met:

Its complement ($\mathbb{R} \setminus A$) is open.

A contains all of its accumulation points.

A contains all of its boundary points.

$A = cl(A)$.

<u>Definition 14.09</u>: A set U is said to be dense in $\mathbb{R}$ iff between any two real numbers, there is an element of U.

<u>Theorem 14.06</u>: $\mathbb{Q}$ is dense in $\mathbb{R}$

<u>Theorem 14.07</u>: $\mathbb{I}$ is dense in $\mathbb{R}$.

Theorem 14.08: $\mathbb{Q}$ is dense in $\mathbb{I}$

Theorem 14.09: $\mathbb{Q}$ is dense in $\mathbb{Q}$

Definition 14.10: A set U is said to be **nowhere dense** in $\mathbb{R}$ iff there is not one interval in $\mathbb{R}$ such that U is dense.

An example of a nowhere dense set is the set of natural numbers ($\mathbb{N}$).

Definition 14.11: A subset S of $\mathbb{R}$ is **perfect** if S = S′.

Definition 14.12: A subset S of $\mathbb{R}$ is **compact** if every open cover of S has a finite subcover of S.

Theorem 14.13: A subset S of $\mathbb{R}$ is **compact** iff it is both closed and bounded.

Definition 14.14: A subset S of $\mathbb{R}$ is **disconnected** $\exists$ two open sets A and B ∍

$S \subseteq A \cup B$ and $A \cap B = \varnothing$.

Definition 14.15: S is **connected** iff it is not disconnected.

Definition 14.16: A subset S of $\mathbb{R}$ is **degenerate** iff it is empty or it is a singleton point set.

Definition 14.17: A subset S of $\mathbb{R}$ is **totally disconnected** iff there is no non-degenerate subset of S that is connected.

One such example of a totally disconnected set is the set of natural numbers.