ADVANCED CALCULUS (REAL ANALYSIS) I
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MORE DEFINITIONS, AXIOMS, AND THEOREMS § 12
HANDOUT 2 ½

Let $U = \mathbb{R}$.

Definition 12.1: A segment (a, b) means $\forall\ x \ni a < x < b$.

Definition 12.2: An interval $[a, b]$ means $\forall\ x \ni a \le x \le b$.

Definition 12.3A: A half segment (or half interval) $[a, b)$ means $\forall\ x \ni a \le x < b$.

Definition 12.3B: A half segment (or half interval) $(a, b]$ means $\forall\ x \ni a < x \le b$.

Definition 12.4: A point, p, is a limit point of the set K means
$\forall\ \varepsilon > 0\ \exists$ a point $x \in K \ni x \in (p - \varepsilon, p + \varepsilon)$ where $p \neq x$.

Alternate Definition 12.4: A point, p, is an accumulation point of the set K means
$\forall\ \varepsilon > 0\ \exists$ a point $x \in K \ni x \in (p - \varepsilon, p + \varepsilon)$ where $p \neq x$.

Notation for Definition 12.4: The set of limit points of the set K is symbolised as K'; so, a point, p, is a limit point of the set K is symbolised as $p \in K'$

Def. 12.5: Let $K \subseteq \mathbb{R}$, K is bounded above iff $\exists$ a point $x \in \mathbb{R} \ni p \le x\ \ \forall\ p \in K$.

Def. 12.6: Let $K \subseteq \mathbb{R}$, K is bounded below iff $\exists$ a point $y \in \mathbb{R} \ni y \le p\ \ \forall\ p \in K$.

Def. 12.7: Let $K \subseteq \mathbb{R}$, K is bounded iff it is bounded above and bounded below.

Def.. 12.8: Let $K \subseteq \mathbb{R}$. Define the set U_K where $U_K = \{m \mid m \ge p\ \ \forall\ \ p \in K\}$ and call it the set of upper bounds of the point-set K.

Def.. 12.9: Let $K \subseteq \mathbb{R}$. Define the set L_K where $L_K = \{q \mid q \le p\ \ \forall\ \ p \in K\}$ and call it the set of lower bounds of the point-set K.

Theorem. 12.1: Let $K \subseteq \mathbb{R} \ni K$ is bounded above.

The set $U_K = \{m \mid m \ge p\ \ \forall\ \ p \in K\}$ is non-empty.

Theorem. 12.2: Let $K \subseteq \mathbb{R} \ni K$ is bounded below.

The set $L_K = \{q \mid q \le p\ \ \forall\ \ p \in K\}$ is non-empty.

Def.. 12.10: Let $K \subseteq \mathbb{R}$. Consider $U_K = \{m \mid m \ge p\ \ \forall\ \ p \in K\}$. Define the point $s \in \mathbb{R}$ to be the supremum of K, denoted $\sup(K)$, (or the least upper bound of K denoted lub(K)) to be the point in U_K such that $s \le m\ \ \forall\ m \in U_K$.

<u>Def.. 12.11</u>: Let $K \subseteq \mathbb{R}$. Consider $U_K = \{m \mid m \geq p \quad \forall \quad p \in K\}$. Define the point $g \in \mathbb{R}$ to be the infinum of K, denoted $\inf(K)$, (or the greatest lower bound of K denoted glb(K)) to be the point in L_K such that $m \leq g \quad \forall\ m \in L_K$.

<u>Axiom of Completeness</u>: Let $K \subseteq \mathbb{R}$, $K \neq \varnothing$, $\wedge$ K be bounded above. Then $\sup(K)$ exists and is obviously real.

<u>Definition 12.12:</u> A point, p, is a boundary point of the set K means $\forall\ \varepsilon > 0$
$\exists$ a point $x \in K \ni x \in (p - \varepsilon, p + \varepsilon) \wedge$
$\exists$ a point $y \in K^c \ni y \in (p - \varepsilon, p + \varepsilon)$.

<u>Notation for Definition 12.12:</u> A point, p, is a boundary point of the set K is symbolised as $p \in \mathrm{bd}(K)$
The set of boundary points of the set K is symbolised as $\mathrm{bd}(K)$

<u>Definition 12.13:</u> A point, p, is an interior point of the set K means $\exists\ \varepsilon > 0 \ni$
$\forall\ x \in (p - \varepsilon, p + \varepsilon) \Rightarrow x \in K$ The set of interior points of the set K is symbolised as $\mathrm{int}(K)$

<u>Definition 12.14:</u> K is open means $\mathrm{bd}(K) \subseteq K^C$.

<u>Definition 12.15:</u> K is closed means $\mathrm{bd}(K) \subseteq K$.

<u>Theorem 12.1</u>: Let $K \subseteq \mathbb{R}$, K is open iff K^c is closed.

<u>Definition 12.16:</u> A point, p, is a point of the closure of the set K means $p \in K \vee p \in K'$
The closure of the set K is symbolised as $\mathrm{cl}(K)$

<u>Definition 12.17</u>: A point, p, is an isolated point of the set K means $\exists\ \varepsilon > 0 \ni$
$\forall\ x \in (p - \varepsilon, p + \varepsilon) \Rightarrow x \notin K$ where $x \neq p$
The set of isolated points of the set K is symbolised as $\mathrm{iso}(K)$

<u>Theorem 12.2</u>: Let $K \subseteq \mathbb{R}$, K is closed iff $K = \mathrm{cl}(K)$.

<u>Theorem 12.3</u>: Let $K \subseteq \mathbb{R}$, K is open iff $K = \mathrm{int}(K)$.

<u>Notation:</u> Let $K \subseteq \mathbb{R}$ and Let $M \subseteq \mathbb{R}$.

A difference set $K - M$ is also written in this class as $K \setminus M = \{x \mid x \in K \wedge x \notin M\}$

<u>Theorem 12.4</u>: Let $K \subseteq \mathbb{R}$, $K = \cup\, \Gamma$ such that $\Gamma = \{A \mid A \text{ is open}\}$. K is open.

Theorem 12.5: Let $K \subseteq \mathbb{R}$, $K = \cup\,\Gamma$ such that $\Gamma = \{A \mid A$ is closed$\}$ and $|\Gamma| < \aleph_0$. K is closed.

Theorem 12.6: Let $K \subseteq \mathbb{R}$, $K = \cap\,\Gamma$ such that $\Gamma = \{A \mid A$ is closed$\}$. K is closed.

Theorem 12.7: Let $K \subseteq \mathbb{R}$, $K = \cap\,\Gamma$ such that $\Gamma = \{A \mid A$ is open$\}$ and $|\Gamma| < \aleph_0$. K is open.