Ms. Andrea Bongco Math 351 Fall 2009 Worksheet 1 Problem 13.

Claim: $|\mathbb{Z}| = \aleph_0$

<u>Proof</u>: Assume the premises.

Let g: $\mathbb{Z} \longrightarrow \mathbb{N}$ where $g(x) = \begin{cases} 2x, & x > 0 \\ -2x+1, & x \leq 0 \end{cases}$ $g \subseteq \mathbb{Z} \times \mathbb{N}$ so g is a relation from $\mathbb{Z}$ to $\mathbb{N}$

which implies dom(g) = $\mathbb{Z}$, cor(g) $\subseteq \mathbb{Z}$, cod(g) = $\mathbb{N}$, and ran(g) $\subseteq \mathbb{N}$

Suppose dom(g) $\neq$ cor(g) since cor(g) $\subseteq$ dom(g) from above this forces cor(g) $\subset$ dom(g),

so $\exists$ x$\in$ dom(g) $\ni$ x $\notin$ cor(g)

Case 1: x > 0.

Since x $\in \mathbb{Z} \wedge$ x > 0 $\Rightarrow$ x $\in \mathbb{N}$ by definition of $\mathbb{N}$

Consider 2x, 2x $\in \mathbb{N}$ Since 2 $\wedge$ x are natural numbers and $\mathbb{N}$ is closed under multiplication

Therefore (x, 2x) $\in$ g

Thus, x $\in$ cor(g) **#!** of x $\notin$ cor(g) $\Rightarrow$ in this case x $\in$ cor(g)

Case 2: x = 0

Consider $-2(0) + 1$

$-2(0) + 1 = 0 + 1$ by lemma 2

$0 + 1 = 1$ by Existence of the Additive Identity Axiom

1$\in \mathbb{N}$

Therefore (0, 1) $\in$ g

Thus, x $\in$ cor(g) **#!** of x $\notin$ cor(g) $\Rightarrow$ in this case x $\in$ cor(g)

Case 3: x < 0

x is negative by Definition of negative real number and x we have x $\in \mathbb{Z}$ from above.

Consider $-2\cdot x + 1$, $-2\cdot x > 0$ Since the product of two negatives is a positive (exercise 11.3 (b) in

§11 Lay book) and since $-2 \in \mathbb{Z}$ and $x \in \mathbb{Z}$, $-2\cdot x \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under multiplication

But $-2\cdot x > 0$ and $-2\cdot x \in \mathbb{Z} \Rightarrow -2\cdot x \in \mathbb{N}$

$1 \in \mathbb{N}$ by the Peano Axioms, so we have $-2\cdot x + 1 \in \mathbb{N}$ since $\mathbb{N}$ is closed under addition.

Therefore $(x, -2\cdot x + 1) \in g$

Thus, $x \in \text{cor}(g)$ **#!** of $x \notin \text{cor}(g) \Rightarrow$ in this case $x \in \text{cor}(g)$

Since all cases are covered, we note $\text{cor}(g) \subseteq \text{dom}(g)$ and $\text{dom}(g) \subseteq \text{cor}(g)$; hence, $\text{cor}(g) = \text{dom}(g)$.

Suppose $(x, y) \in g$ and $(x, z) \in g$ such that $y \neq z$

Case 1: $x > 0$

$y = 2x$ and $z = 2x \Rightarrow y = z$ by transitivity of equality **#!** of $y \neq z$. Thus in this case $y = z$.

Case 2: $x = 0$

$y = -2\cdot x + 1$ and $z = -2\cdot x + 1 \Rightarrow y = z$ by transitivity of equality. So we have $y \neq z \wedge y = z$ by adjunction **#!** of the law of the excluded middle so the supposition is false and we must conclude that g maps x to a unique y.

Therefore by the definition of function from Math 224, g is a well defined function.

Suppose g is not an injection from $\mathbb{Z}$ to $\mathbb{N}$

Thus, we suppose $\exists$ $(m, y) \in g$ and $(n, y) \in g \ni m \neq n$. $m \wedge n$ are integers since $g \subseteq \mathbb{Z} \times \mathbb{N}$

Case 1: y is an even natural number

$y = 2\cdot m$ and $y = 2\cdot n$ so $2\cdot m = 2\cdot n$ by Transitivity of Equality

Now, $\frac{1}{2}(2m) = \frac{1}{2}(2n)$ by Preservation of equality under multiplication

$(\frac{1}{2}\cdot 2)\, m = (\frac{1}{2}\cdot 2)\, n$ by Associative Multiplication Axiom

$1m = 1n$ by the existence of multiplicative inverse Axiom

$m = n$ by the existence of multiplicative identity Axiom **#!** of $m \neq n$

Case 2: y is an odd natural number

$y = 2\cdot m + 1$ and $y = 2\cdot n + 1$ where $m \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$.

so $2\cdot m+1 = 2\cdot n+1$ by Transitivity of Equality

Now $(2\cdot m+1)+(-1) = (2\cdot n+1)+(-1)$ by Preservation of Equality under Addition

Hence $2 \cdot m + (1 + (-1)) = 2 \cdot n + (1 + (-1))$ by Associativity of Addition Axiom

$\Rightarrow 2 \cdot m + 0 = 2 \cdot n + 0$ by Existence of Additive Inverse

$\Rightarrow 2 \cdot m = 2 \cdot n$ by Existence of Additive Identity

$\Rightarrow \frac{1}{2}(2 \cdot m) = \frac{1}{2}(2 \cdot n)$ by Preservation of equality under Multiplication

$\Rightarrow (\frac{1}{2} \cdot 2)\, m = (\frac{1}{2} \cdot 2)\, n$ by Associativity of Multiplication

$\Rightarrow 1 \cdot m = 1 \cdot n$ by Existence of Multiplicative Inverse

$\Rightarrow m = n$ by Existence of Multiplicative Inverse Axiom **#! #!** of $m \neq n$

So, $\nexists$ $(m, y) \in g$ and $(n, y) \in g$ where $m \neq n \Rightarrow$ g is an injective function.

Suppose g is not a surjection from $\mathbb{Z}$ to $\mathbb{N}$

This means we (under the supposition) have $cod(g) \neq ran(g)$. We have $ran(g) \subseteq cod(g)$ since g is a relation from the integers to the natural numbers therefore we must (under the supposition) note $cod(g) \not\subseteq ran(g)$

i.e. $\exists\, p \in cod(g) \ni p \notin ran(g)$. Since $\in cod(g)$ that implies that p is a natural number.

Case 1: p is odd, i.e. p is of the form $2 \cdot n - 1 \ni n \in \mathbb{N}$ (definition of odd natural number)

$p = 2 \cdot n - 1$ for some natural number n.

We know $n = 1$ by Definition of $\mathbb{N}$

Sub Case 1: Suppose $p = 1$. $p = 2 \cdot n - 1$ which forces $n = 1$. Notice we can consider the

integer $r = 0$; and, consider $2 \cdot r + 1 = 2(0) + 1 = 1$; so, $(0, 1) = (r, p)$ and so $(r, p) \in g$ **#!** of

$p \notin ran(g)$ since $(r, p) \in g$ MAKES $p \in ran(g)$

Sub Case 2: $p > 1$, $n \in \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{Z}$ so $p \in \mathbb{Z}$

$p = 2 \cdot m - 1$ where $m \in \mathbb{N}$ (definition of odd natural number)

$2 \cdot m - 1 = 2 \cdot m + 0 - 1$ by the axiom of the additive identity

$2 \cdot m - 1 = 2 \cdot m + 2 - 2 - 1$ by the axiom of the additive inverse

$2 \cdot m - 1 = 2 \cdot m - 2 + 2 - 1$ by the axiom of commutativity of addition

$2 \cdot m - 1 = (2 \cdot m - 2) + 2 - 1$ by the axiom of associativity law of addition

$2 \cdot m - 1 = (2 \cdot (m - 1)) + 2 - 1$ by the axiom of distributive of multiplication over addition

$2 \cdot m - 1 = (2 \cdot (m - 1)) + 1$ by arithmetic $(2 - 1 = 1)$ and substitution

$p = (2 \cdot (m - 1)) + 1$ by transitivity of equality and since $m \in \mathbb{N}$ $(m - 1) \in \mathbb{N}^*$

so let us rename the element $(m - 1)$, w.

$p = 2 \cdot w + 1$ by substitution where $w \in \mathbb{N}^*$

So, $p = 1 \cdot (2 \cdot w + 1)$ by substitution (and we know $w \in \mathbb{N}^*$)

$p = (-1)(-1)2 \cdot w + 1$ by Lemma 3

$p = ((-1)2)((-1)w) + 1$ by the Axioms of Associativity and Commutativity of $\times$

$p = ((-2)(-w)) + 1$ by Lemma 4

Since $w \in \mathbb{N}^*$ we know $-w$ exists and is an integer by the axiom of the additive

inverse so let us rename $-w$, q. We know q is an integer that is less than or equal to 0.

Therefore $p = -2 \cdot w + 1$ where $q \in \mathbb{Z} \wedge q \leq 0$

But we therefore have $q \in dom(g)$ and $(q, p) \in g$

So, $p \in ran(g)$ **#!**

Case 2: p is an even natural number, i.e. p is of the form 2t for some $t \in \mathbb{N}$ $(p > 0, 2 > 0$ forces $t > 0)$

However, $p = 2t \Rightarrow (t, p) \in g$

$\Rightarrow p \in ran(g)$ **#!**

Thus, $p \in cod(g) \Rightarrow p \in ran(g)$ and we have $cod(g) \subseteq ran(g)$

Hence we have $ran(g) = cod(g) \Rightarrow$ g is a surjection.

Therefore, g is a bijection from $\mathbb{Z}$ to $\mathbb{N}$ and that implies the cardinality of the integers is equal to the cardinality of the natural numbers by Definition of Equinumerability of sets.

Hence, in notational writing we have $|\mathbb{Z}| = |\mathbb{N}|$ (or in the language of sets the integers are denumerable)

Thus $|\mathbb{Z}| = \aleph_0$ by the definition of cardinality of $\mathbb{N}$ (what $\aleph_0$ means)

QED