Real Analysis Notes 2021 – 2022

Appendix A: Set Theory

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based on the notes of **Dr. Coke Reed; Dr. R. L. Moore;**
Dr. J. W. Neuberger; Dr. W. S. Mahavier; Dr. M. Smith;
Dr. D. Riddle; Dr. H. Sharp; and, **Dr. W. T. Mahavier**
along with others I have known or interacted with since my first Real Analysis course
in 1979 **AD (not BC it just seems to be the case).**

Math 351 - Math 352

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1. Review of Background Set Theory

Because of what occurred from the Spring of 2020 through the Spring of 2021; we need to begin with some material from Math 224; namely, the theory of functions. They are built from product sets and relations so the material on product sets and relations you need to read and understand. We will be discussing said to review then march into the theory of function for a week or so.

1.1. **Product Sets: Computational or Graphing.** We started this section of the course building from sets to the more general product sets.
Recall: Let U be a well defined universe. Let V be another well defined universe. Let U be a well defined universe. Let A be a set whose points, p are elements from U. Let B be a set whose points, b are elements of V. We defined the universe $W = U \times V$.

Definition 1.1. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The product set A with B is the set $A \times B = \{(a, b) : a \in A, b \in B\}$.

Let $W = \mathbb{R} \times \mathbb{R}$ Let $A, B, C, D, E, F, G, H \wedge J$ be non-empty sets such that

$A = \mathbb{N}_4$	$B = \mathbb{Z}$	$C = [1, 4)$
$D = (1, 4)$	$E = \{4, 9\}$	$F = [1, 7]$
$G = (1, 7]$	$H = \{1, 4, 6, 7, 8\}$	$J = \{x \mid 1 < x \leq 4\}$

Computational Exercises:

Exercise 1.2. Find the set (list out all the elements of): $A \times E$

Exercise 1.3. Find the set (list out all the elements of): $A \times H$

Exercise 1.4. Find the set (list out all the elements of): $E \times A$

Exercise 1.5. Find the set (list out all the elements of): $A \times (E \cap H)$

Exercise 1.6. Graph the set $A \times E$ in the Cartesian plane.

Exercise 1.7. Graph the set $A \times H$ in the Cartesian plane.

Exercise 1.8. Graph the set $E \times A$ in the Cartesian plane.

Exercise 1.9. Graph the set $A \times (E \cap H)$ in the Cartesian plane.

Exercise 1.10. Graph the set $C \times G$ in the Cartesian plane.

Exercise 1.11. Graph the set $C \times F$ in the Cartesian plane.

Exercise 1.12. Graph the set $D \times G$ in the Cartesian plane.

Exercise 1.13. Graph the set $J \times G$ in the Cartesian plane.

Definition 1.14. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$, $C \subseteq U$ whilst $B \subseteq V$ and $D \subseteq V$.

(1) The product set $(A \times B) \cup (C \times D)$ is the set $\{(m, n) : (m, n) \in A \times B \vee (m, n) \in C \times D\}$.

(2) The product set $(A \times B) \cap (C \times D)$ is the set $\{(m, n) : (m, n) \in A \times B \wedge (m, n) \in C \times D\}$.
(3) The product set $(A \cap C) \times (B \cap D)$ is the set $\{(m, n) : m \in A \cap C \wedge n \in B \cap D\}$.
(4) The product set $(A \cup C) \times (B \cup D)$ is the set $\{(m, n) : m \in A \cup C \wedge n \in B \cup D\}$.

1.2. **Product Sets: Computational or Graphing Continued.** We started this section of the course building from sets to the more general product sets.
Recall: Let U be a well defined universe. Let V be the well defined universe. Let U be a well defined universe. Let A be a set whose points, p are elements from U. Let B be a set whose points, b are elements of V. We defined the universe $W = U \times V$.

Definition 1.15. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The product set A with B is the set $A \times B = \{(a, b) : a \in A, b \in B\}$.

For all exercises let $U = \mathbb{R}$, $V = \mathbb{R} \times \mathbb{R}$ Let $A, B, C, D, E, F, G, H \wedge J$ be non-empty sets from U such that

$A = \mathbb{N}_4$	$B = \mathbb{Z}$	$C = [1, 4)$
$D = (2, 5)$	$E = \{4, 9\}$	$F = [1, 7]$
$G = (1, 7]$	$H = \{1, 4, 6, 7, 8\}$	$J = \{x \vert 2 < x \leq 3\}$

Exercise 1.16.
1. Find the set (list out all the elements of): $A \times E$
2. Find the set (list out all the elements of): $A \times H$
3. Find the set (list out all the elements of): $E \times A$
4. Find the set (list out all the elements of): $A \times (E \cap H$

Definition 1.17. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$, $C \subseteq U$ whilst $B \subseteq V$ and $D \subseteq V$.
(1) The product set $(A \times B) \cup (C \times D)$ is the set $\{(m, n) : (m, n) \in A \times B \vee (m, n) \in C \times D\}$.
(2) The product set $(A \times B) \cap (C \times D)$ is the set $\{(m, n) : (m, n) \in A \times B \wedge (m, n) \in C \times D\}$.
(3) The product set $(A \cap C) \times (B \cap D)$ is the set $\{(m, n) : m \in A \cap C \wedge n \in B \cap D\}$.
(4) The product set $(A \cup C) \times (B \cup D)$ is the set $\{(m, n) : m \in A \cup C \wedge n \in B \cup D\}$.

Exercise 1.18. Graphing the following product sets (if possible; if something does not exist state why):
1. Draw $A \times B$.
2. Draw $A \times C$.
3. Draw $A \times D$.
4. Draw $A \times A$.
5. Draw $A \times E$.
6. Draw $A \times F$.
7. Draw $J \times G$.
8. Draw $G \times J$.
9. Draw $(D \times F)$.
10. Draw $(D \times E) \cup (F \times G)$
11. Draw $(C \times D) \cup (F \times G)$
12. Draw $(D \times C) \cap (F \times G)$
13. Draw $(D^c \times F)$
13. Draw $(F - E) \times F$

Exercise 1.19. So far this semester we have considered theorems about points, p, sets, A, power sets, $\mathcal{P}(A)$, and collections, Ω. Now we have some claims about product sets to consider; but, it is always helpful to draw a picture or pictures to understand a claim. e.g.:

Claim 1.20. Let U be a well-defined universe and $V = U \times U$. Let $W, X, Y, \wedge Z$ be non-empty sets of elements from the universe U. It is the case that $(W \times X) \cap (Y \times Z) \subseteq (W \cap Y) \times (X \cap Z)$.

What we do is think about the claim and try to figure out if it is true or not. We draw different scenarios that fulfill the premises and see if the conclusion follows (how many examples, etc. is not well defined - do as **many** as it takes to visualise the scenario).

Use the same universe and sets from exercise 12.1 and 12.2:
Draw $(A \times F) \cap (C \times D)$ and $(A \cap C) \times (F \cap D)$.
Do these seem to indicate the claim is true or not?

Draw $(E \times B) \cap (C \times J)$ and $(E \cap C) \times (B \cap J)$.
Do these seem to indicate the claim is true or not?

Draw $(J \times G) \cap (C \times A)$ and $(J \cap G) \times (C \cap A)$.
Do these seem to indicate the claim is true or not?

1.2.1. *Prove the Following Theorem.* Recall: Let U be a well defined universe. Let V be the well defined universe, $\mathcal{P}(U)$. Let U be a well defined universe. Let A be a set whose points, p are elements from U. Let B be a set whose points, b are elements of V. We defined the universe $W = U \times V$.

Definition 1.21. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The product set A with B is the set $A \times B = \{(a, b) : a \in A, b \in B\}$.

Theorem 1.22. Let $V = U \times U$ Let $A, B, C, \wedge D$ be non-empty sets of elements from U. It is the case that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

1.3. **Prove or Disprove The Following Claims.**

Claim 1.23. Let U be a well defined universe. Let $V = U \times U$ Let $A, B, C, \wedge D$ be non-empty sets of elements from U.
It is the case that $(A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$.

Claim 1.24. Let U be a well defined universe. Let $V = U \times U$ Let $A, B, C, \wedge D$ be non-empty sets of elements from U.
It is the case that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$.

Claim 1.25. Let U be a well defined universe. Let $V = U \times U$ Let $A, B, C, \wedge D$ be non-empty sets of elements from U.
It is the case that $(A \times B) - (C \times D) \subseteq (A - C) \times (B - D)$.

Claim 1.26. Let U be a well defined universe. Let $V = U \times U$ Let $A, B, C, \wedge D$ be non-empty sets of elements from U.
It is the case that $(A - B) \times (C - D) \subseteq (A \times C) - (B \times D)$.

Claim 1.27. Let U be a well defined universe. Let $V = U \times U$ Let $A \wedge B$ be non-empty sets of elements from U.
It is the case that $(A \times B)^C \subseteq A^C \times B^C$.

Claim 1.28. Let U be a well defined universe. Let $V = U \times U$ Let $A \wedge\ B$ be non-empty sets of elements from U.
It is the case that $(A \times B)^C \supseteq A^C \times B^C$.

1.4. A Pair of Thought Provoking Questions.

False Claim 1.29. Let U be a well defined universe. Let $V = U \times U$ Let $A, B \wedge\ C$ be non-empty sets of elements from U.
It is the case that $A \cup B \times C = A \cup B \times B \cup C$.

Why is the preceding claim a nonsensical claim?

False Claim 1.30. Let U be a well defined universe. Let $V = U \times U$ Let $A, B \wedge\ C$ be non-empty sets of elements from U.
It is the case that $A \cup (B \times C) = (A \cup B) \times (B \cup C)$.

Why is the preceding claim *still* a nonsensical claim?

1.5. Relations Part 1.

We started this section of the course building from sets to the more general product sets.
Recall: Let U be a well defined universe. Let V be the well defined universe, $\mathcal{P}(U)$. Let A be a set.
We considered theorems on sets, power sets, $\mathcal{P}(A)$, and sub-collections of power sets, Ω.
Then we considered Let U be a well defined universe. Let V be the well defined universe.
Let A be a set which is a subset of U. Let B be a set whose points are elements of V. We defined the universe $W = U \times V$.

1.6. Relations.

Definition 1.31. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The product set A 'cross' B is the set
$A \times B = \{(a, b) : a \in A, b \in B\}$.

Let R be any subset of $A \times B$:
$R \subseteq A \times B$. R is called a relation. Big whoop!
We found the subject of relations in and by themselves was not too intriguing; but, there are sets defined by the relation.

Definition 1.32. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $M \subseteq A \times B$. The **corange** of the relation M from A to B is the set A.

Notation 1.33. The **corange** of the relation M from A to B is denoted as $cor(M)$.

Definition 1.34. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $M \subseteq A \times B$. The **codomain** of the relation M from A to B is the set B.

Notation 1.35. The **codomain** of the relation M from A to B is denoted as $cod(M)$.

Definition 1.36. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $M \subseteq A \times B$. The **domain** of the relation M from A to B is the set $\{x : (x, y) \in M\}$.

Notation 1.37. The **domain** of the relation M from A to B is denoted as $dom(M)$.

Definition 1.38. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $M \subseteq A \times B$. The **range** of the relation M from A to B is the set $\{y : (x, y) \in M\}$.

Notation 1.39. The **range** of the relation M from A to B is denoted as $ran(M)$

1.7. **Exercises:**

Exercise 1.40. Let the universe $W = \mathbb{N} \times \mathbb{N}$.
Consider $A = \{1, 2, 3, 4\}$. Consider the relation $R \subseteq A \times A$ where
$R = \{(1, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\}$
1.1. Find $cor(R)$. 2.1 Find $dom(R)$. 3.1 Find $ran(R)$ 4.1 Find $cod(R)$

Exercise 1.41. Let the universe $W = \mathbb{Z} \times \mathbb{N}$. Consider the relation $R \subseteq A \times B$.
Consider $A = \{-3, -1, 0, 2, 3, 4\}$, $B = \mathbb{N}_5$ Let $R = \{(x, y) : x = -y \vee x = y\}$
1.2. Find $cor(R)$. 2.2 Find $cod(R)$. 3.2 Find $dom(R)$ 4.2 Find $ran(R)$

Exercise 1.42. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Consider $A = \mathbb{N}$, $B = \{p : p \in \mathbb{N} \wedge p^2 \in \mathbb{N}\}$
Consider the relation $T \subseteq A \times B$ where $T = \{(p, p^2) : p \leq 20\}$
1.3. Express T in list form (write out all the points in T).
2.3. Find $cor(T)$. 3.3 Find $cod(T)$. 4.3 Find $dom(T)$ 5.3 Find $ran(T)$

Exercise 1.43. Let the universe $W = \mathbb{N} \times \mathbb{N}$.
Consider $A = \{1, 2, 3, 4\}$. Consider the relation $R \subseteq A \times A$ where
$R = \{(1, 1), (1, 2), (2, 4), (1, 3), (2, 2), (3, 3), (4, 4)\}$
1.4. Find $cor(R)$. 2.4. Find $cod(R)$. 3.4. Find $dom(R)$ 4.4. Find $ran(R)$

Exercise 1.44. Let the universe $W = \mathbb{N} \times \mathbb{N}$.
Consider $A = \mathbb{N}_{12}$. Consider the relation $K \subseteq A \times A$ where $K = \{(a, b) | b \text{ is divisible by } a\}$
1.5. Express K in list form (write out all the points in K).
2.5. Find $cor(K)$. 3.5 Find $dom(K)$. 4.5 Find $ran(K)$ 5.5 Find $cod(K)$

1.8. **Relations Part 2 (with an Introductory Note on Partial Orders and Equivalence Relations).** We started this section of the course building from sets to the more general product sets.

Recall: Let U be a well defined universe. Let V be the well defined universe, $\mathcal{P}(U)$. Let A be a set. We considered theorems on sets, power sets, $\mathcal{P}(A)$, and sub-collections of power sets, Ω.

Then we considered Let U be a well defined universe. Let V be the well defined universe. Let A be a set which is a subset of U.
Let B be a set whose points are elements of V. We defined the universe $W = U \times V$.

Definition 1.45. Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$.
Let R be any subset of $A \times B$ ($R \subseteq A \times B$). R is called a relation.

Example 1.46. Since the relation M from A to B is the set $M \subseteq \{(a, b) : a \in A, b \in B\}$ it can be whatever.
Let the universe $W = U \times V$ be defined from the well defined universes $U = \mathbb{N}$ and $V = \mathbb{N}_{10}$.
Let $A = \mathbb{N}_2$ and let $B = \mathbb{N}_1$.
Notice there are not a ton of relations between A and B.
The set $M_0 = \emptyset$ is a relation from A to B.
The set $M_1 = \{(1, 1)\}$ is a relation from A to B.
The set $M_2 = \{(2, 1)\}$ is a relation from A to B.
The set $M_3 = \{(1, 1), (2, 1)\}$ is a relation from A to B.
The set $M_4 = \{(1, 1), (2, 1), (2, 2)\}$ is NOT relation from A to B; it is a relation from U to V but not from A to B.
We found the subject of relations in and by themselves was not too intriguing.

1.9. **Relations on a Set.** So, we consider relations on a set.

Definition 1.47. Let the universe $W = U \times U$ be defined from the well-defined universe U such that $A \subseteq U$.
The **relation S from A to A** (S is on A) is the set $S \subseteq \{(a, b) : a \in A, b \in A\}$.

Oddly, this creates more interesting ideas.

Definition 1.48. : Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A. R is said to be **reflexive** if for every $x \in A, (x, x) \in R$ and $\exists\, x \in A$ (note there must exist an x such that $(x, x) \in R$, this is to ensure that $\emptyset$ does not vacuously hold as reflexive).

Definition 1.49. Let the universe $W = U \times U$ be defined from the well defined universe U such that A $\subseteq$ U. Consider the relation R on A. R is said to be **symmetric** if for every $(x.y) \in R \Rightarrow (y, x) \in R$.

Definition 1.50. Let the universe $W = U \times U$ be defined from the well defined universe U such that A $\subseteq$ U. Consider the relation R on A. R is said to be **antisymmetric** if $(x, y) \in R$ where $x \neq y \;\Rightarrow\; (y, x) \notin R$.

Alternate Definition for Antisymmetry: *Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A. R is said to be* **antisymmetric**
if $(x, y) \in R \;\wedge\; (y, x) \in R \;\Rightarrow\; x = y$.

Exercise 1.51. Show that the two definitions for antisymmetry are logically equivalent.

Definition 1.52. : Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A. R is said to be **transitive** if $(x.y) \in R \;\wedge\; (y, z) \in R \;\Rightarrow\; (x, z) \in R$.

Notice this is **not** the same as saying the following: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A. $(x, y) \in R \Rightarrow \exists\, z \ni (x, z) \in R \wedge (z, y) \in R$.

1.10. **Exercises:**

Exercise 1.53. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation R on A. Consider A = {1, 2, 3, 4} Let R = {(1, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)}

(1) Determine if R is reflexive or not. Justify.
(2) Determine if R is symmetric or not. Justify.
(3) Determine if R is antisymmetric or not. Justify.
(4) Determine if R is transitive or not. Justify.

Exercise 1.54. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A. Consider A = {1, 2, 3, 4} Let S = {(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)}

(1) Determine if S is reflexive or not. Justify.
(2) Determine if S is symmetric or not. Justify.
(3) Determine if S is antisymmetric or not. Justify.
(4) Determine if S is transitive or not. Justify.

Exercise 1.55. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation T on A. Consider A = {1, 2, 3, 4} Let T = {(1, 1), (1, 2), (1, 3), (2, 3), (2, 2), (1, 4), (3, 3), (4, 4)}

(1) Determine if T is reflexive or not. Justify.
(2) Determine if T is symmetric or not. Justify.
(3) Determine if T is antisymmetric or not. Justify.
(4) Determine if T is transitive or not. Justify.

Exercise 1.56. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation W on A. Consider A = {1, 2, 3, 4} Let W = {(1, 1), (1, 2), (1, 3), (2, 4), (2, 2), (1, 4), (3, 3), (4, 4)}

(1) Determine if W is reflexive or not. Justify.
(2) Determine if W is symmetric or not. Justify.
(3) Determine if W is antisymmetric or not. Justify.
(4) Determine if W is transitive or not. Justify.

1.11. **Partial Orders and Equivalence Relations.**

Definition 1.57. Let the universe $W = U \times U$ be defined from the well defined universe U such that A ⊆ U. Let the relation be R on the set A. If R is reflexive, symmetric, and transitive, then R is called an **equivalence relation on A**.

Example 1.58. Let the universe Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider $A = \{1, 2, 3, 4\}$.
Consider the relation S on A where $S = \{(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)\}$
S is an equivalence relation on A.

Definition 1.59. Let the universe $W = U \times U$ be defined from the well defined universe U such that A ⊆ U. Let the relation be R on the set A. If R is reflexive, antisymmetric, and transitive, then R is called an **partial order on A**.

Example 1.60. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider $A = \{1, 2, 3, 4\}$. Consider the relation M on A where $M = \{(1, 1), (2, 2), (3, 2), (2, 4), (3, 4), (3, 3), (4, 4)\}$ M is a partial order on A.

1.12. **Equivalence Classes and Partitions.**

Definition 1.61. Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let the relation be R on the set A. If R is an equivalence relation on A, then R induces a subset of $\mathcal{P}(A)$ [a collection], called the collection of equivalence classes of A induced by R.
An equivalence class of the element a ∈ A from R is denoted as $[a]_R$
$[a]_R = \{b \in A : (a, b) \in R\}$.

Example 1.62. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A.
Consider $A = \{1, 2, 3, 4\}$ Let $S = \{(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)\}$
$[1]_S = \{1\}$.
$[2]_S = \{2, 3\} = [3]_S$.
$[4]_S = \{4\}$.
Notice that $\Psi = \{[1]_S, [2]_S, [4]_S\} = \{\{1\}, \{2, 3\}, \{4\}\} \subseteq \mathcal{P}(A)$.
So, we are again considering power sets when discussing partitions or equivalence classes.

Definition 1.63. Let the universe U be defined and let the set A be well defined such that $A \subseteq U$.
Let the collection $\Pi \subseteq \mathcal{P}(A)$. Π is called a **partition of A** iff:
(1) $\forall\ M \in \Pi\ \wedge\ \forall\ N \in \Pi$, it is the case that $M \cap N = \emptyset$; $\wedge$,
(2) $\cup\Pi = A$.
Condition (1) means the sets in Π are pair-wise disjoint or they are said to be mutually exclusive; and,
Condition (2) means the collection Π is exhaustive.

Theorem 1.64. Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let the R be an equivalence relation on the set A. The collection of equivalence classes of A induced by R partitions A.

Theorem 1.65. Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let Ω be a partition of A. Ω induces an equivalence relation on A which is typically denoted as $A \backslash \Omega$.[1]

Example 1.66. (An Equivalence Relation creates a Partition)
Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A. Consider A = {1, 2, 3, 4} Let S = {(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)}
$[1]_S = \{1\}.[2]_S = \{2, 3\} = [3]_S.[4]_S = \{4\}$.
Notice that $\Psi = \{[1]_S, [2]_S, [4]_S\}$ partitions A.

[1]I will use the notation ω for the equivalence relation induced by the partition Ω (upper case versus lower case).

Example 1.67. (A Partition creates an Equivalence Relation)
Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$.
Consider the partition of $A, \Phi = \{\{1, 2\}, \{3, 4\}\}$.
Notice it induces the equivalence classes $[1]_\Phi = [2]_\Phi$ and $[3]_\Phi = [4]_\Phi$
which when we 'backtrack' induces the equivalence relation, call it ϕ where
$\phi = \{(1, 1), (2, 2), (1, 2), (2, 1), (3, 3), (4, 4), (3, 4), (4, 3)\}$

I say, whoop dee doo. I love partial orders (next worksheet); so, we will be studying those more than the equivalence relations, classes, and partitions.

1.13. **Functions Preliminaries.** We started this section of the course building from sets to the more general product sets. Then came relations, equivalence relations, equivalence classes, partial orders, linear orders, and well ordering.

Definition 1.68. Let U be a well defined universe. We defined the universe $W = U \times U$. Let A be any subset of U. A is **well ordered under an order**, call it $\preceq$ if and only if the order is a linear order and for any subset, B of A, there is a first element of B under the order $\preceq$.

Axiom 1.69. The Axiom of Choice. Let U be a well defined universe. Given any non-empty collection, Γ, of subsets of U whose members are pair-wise disjoint non-empty sets, there exists a set B consisting of exactly one element taken from each sets belonging to Γ.

Axiom 1.70. The Well Ordering Principle. Let U be a well defined universe and M be a non-empty set. There exists a well-ordering of the set M.

Definition 1.71. Let U be a well-defined universe. Let V be a well-defined universe. We defined the universe $W = U \times V$. Let D and C be sets such that $D \subseteq U$ and $C \subseteq V$. Further, let f be a relation from D to C. f is a **function** from D to C *iff*:

(1) $D \neq \emptyset$;
(2) $dom(D) = cor(D)$;
(3) $(x, y) \in f \ \wedge \ (x, z) \in f \implies y = z$.

Notation 1.72. Suppose U be a well-defined universe; V be a well-defined universe; and, consider the universe $W = U \times V$. Let D and C be sets such that $D \subseteq U$ and $C \subseteq V$. Further, let f be a well-defined function from D to C. We write $f : D \longrightarrow C$ as expressing such.

One may write something down in notation that makes it seem we are discussing a well-defined function, but in actuality it is not.

Example 1.73. Suppose $U = \mathbb{R}$ and $W = \mathbb{R} \times \mathbb{R}$. Let $D = \{1, 2\}$ and $C = \{2, 3, 5\}$ Further, let $g = \{(1, 3), (1, 5), (2, 2)\}$. A person can write $g : D \longrightarrow C$ but it is not a function (explain why).

For all exercises let $U = \mathbb{R}$ and $W = \mathbb{R} \times \mathbb{R}$.

Exercise 1.74. Explain why someone claiming $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = \sqrt{x}$ is a function is incorrect.

Exercise 1.75. Consider $g : \mathbb{N} \longrightarrow \mathbb{R}$ such that $g(x) = \sqrt{x}$. Is it a function or is it not –justify why you opine such.

Exercise 1.76. Consider $h : \mathbb{N} \longrightarrow \mathbb{Z}$ such that $h(x) = \sqrt{x}$. Is it a function or is it not –justify why you opine such.

Exercise 1.77. Consider $j : \{7\} \longrightarrow \mathbb{R}$ such that $j(x) = \sqrt{x}$. Is it a function or is it not –justify why you opine such.

Exercise 1.78. Consider $k : \mathbb{N} \longrightarrow \mathbb{N}$ such that $k(x) = \sqrt{x}$. Is it a function or is it not –justify why you opine such.

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Definition 1.79. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$. Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **injective** *iff* $(x, y) \in f \ \wedge \ (w, y) \in f \implies x = w$.

Notation 1.80. The symbol for f is a well defined injection from D to C is $f : D \rightarrowtail C$ or $D \overset{f}{\rightarrowtail} C$

Definition 1.81. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$. Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **surjective** *iff* $ran(f) = cod(f)$.

Notation 1.82. The symbol for f is a well defined surjection from D to C is $f : D \twoheadrightarrow C$ or $D \overset{f}{\twoheadrightarrow} C$

Definition 1.83. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$. Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **bijective** *iff* it is injective and surjective.

Notation 1.84. The symbol for f is a well defined bijection from D to C is $f : D \rightarrowtail\!\!\!\!\rightarrow C$ or $D \overset{f}{\rightarrowtail\!\!\!\!\rightarrow} C$

Theorem 1.85. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$. Let $D \subseteq U$ and $C \subseteq W$.
Consider the bijective function, f, from D to C.
The relation f^{-1} from C to D is a bijective function also.

Definition 1.86. A function which satisfies the previous theorem is said to be f an **invertible** function.

Definition 1.87. Let U be a well-defined universe. Let $V = U \times W$. Let $V = U \times \mathbb{R}$. Let $D \subseteq U$ and $C \subseteq \mathbb{R}$.
Consider the well defined function $f \subseteq D \times C$. f is called a **real-valued function** simply because the codomain is a subset of $\mathbb{R}$.

Definition 1.88. Let $V = \mathbb{R} \times \mathbb{R}$. Let $D = \mathbb{N}$ and $C \subseteq \mathbb{R}$.
Consider the well defined function $f \subseteq D \times C$. f is called a **sequence** simply because the domain is $\mathbb{N}$.

1.14. **Functions Part 1.**

Definition 1.89. Let U be a well-defined universe and W be a well-defined universe. Let $V_1 = U \times W$. Let $D \subseteq U$ and $C \subseteq W$. Consider the relation, R, from D to C.
The relation R^{-1}, from C to D is the set of all points $\{(y, x) : (x, y)$ is a point of $R\}$. So, R^{-1} is a subset of $C \times D$ whilst the universe for the inverse relation is $V_2 = W \times U$

Definition 1.90. Let U be a well-defined universe, V be a well-defined universe, and W be a well-defined universe. Let $Y_1 = U \times V$ and $Y_2 = V \times W$. Let $D \subseteq U$ and $C_1 \subseteq V$; $C_2 \subseteq V$; $C_1 \cap C_2 \neq \emptyset$; and, $E \subseteq W$. Let the relation, R, be well-defined from D to C_1 and let the relation, S, be well-defined from C_2 to E.
The relation M, from D to E is the set of all points $\{(x, z) : \exists\ (x, y)$ that is a point of R and $\exists\ (y, z)$ that is a point of $S\}$.

Notation 1.91. So, from the definition M is a subset of $D \times E$ and is denoted as $M = S \circ R$

Notice in the definition the $M = S \circ R$ requires there to be 'some agreement' in the middle sets (C_1 and C_2) so there needs to be at least one point p in $C_1 \cap C_2$ so that there is an abscissa x in D so that $(x, p) \in R$ and an ordinate y in E so that $(p, y) \in S$. If $C_1 \cap C_2 = \emptyset$, then the composition is $\emptyset$ and we say the composition is 'trivial.'

Example 1.92. Let $U = V = W = \mathbb{R}$ and let $D = \mathbb{N}_5$ and $C = \mathbb{N}_3$ and $E = \mathbb{N}_4$.
Define $R \subseteq D \times C$ such that $R = \{(1, 2), (2, 3), (4, 1), (5, 1)\}$
Define $S \subseteq C \times E$ such that $S = \{(3, 3), (3, 4), (2, 1)\}$
So, $M = S \circ R$ is $\{(2, 3), (2, 4)\}$.
What of $R \circ S$? It is $\{(2, 2)\}$.

Example 1.93. Let $U = V = W = \mathbb{R}$. Let $D = \mathbb{N}_5$ and $L = \{0, 1\}$ and $E = \mathbb{N}_4$.
Define $A \subseteq D \times L$ such that $A = \{(1, 0), (2, 0)\}$
Define $B \subseteq L \times E$ such that $B = \{(1, 4), (1, 3)\}$
So, $A \circ B$ is $\emptyset$ and $B \circ A$ is $\emptyset$. I would say that is quite trivial.

Now, to perhaps the most important concept, I opine, in all of mathematics.

Definition 1.94. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$.
Let $D \subseteq U$ and $C \subseteq W$. Consider the relation, f, from D to C. f is a **well-defined function** from D to C *iff* $dom(f) \neq \emptyset$; $cor(f) = dom(f)$; and, $(x, p) \in f \wedge (x, q) \in f$ implies that $p = q$.

Definition 1.95. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$.
Let $D \subseteq U$ and $C \subseteq W$. Consider the function, f, from D to C.
f is **injective** *iff* $(x, y) \in f\ \wedge\ (w, y) \in f \implies x = w$.

Definition 1.96. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$.
Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **surjective** *iff* $ran(f) = cod(f)$.

Definition 1.97. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$.
Let $D \subseteq U$ and $C \subseteq W$.
Consider the function, f, from D to C.
f is **bijective** *iff* it is injective and surjective.

Theorem 1.98. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$.
Let $D \subseteq U$ and $C \subseteq W$.
Consider the bijective function, f, from D to C.
The relation f^{-1} from C to D is a bijective function also.

Definition 1.99. A function which satisfies the previous theorem is said to be f an **invertible** function.

Prove or disprove each claim.

Claim 1.100. Consider the relation $g \subseteq [0,\infty) \times \mathbb{R}$ such that $g(x) = x^2$. g is a well defined function.

Claim 1.101. Consider the relation $h \subseteq \mathbb{R} \times [0,\infty)$ such that $g(x) = x^3$. h is a well defined function.

1.15. **Questions Part A.**

Exercise 1.102. Let $U = V = W = \mathbb{R}$; $Y_1 = \mathbb{R} \times \mathbb{R}$; and, $Y_2 = \mathbb{R} \times \mathbb{R}$.
$A = \mathbb{N}_5$ and $B = \{-1, 0, 3, 5, 6\}$; and $C = \{1, 2, 4\}$.
Define $R \subseteq A \times B$ such that $R = \{(1,-1),(2,-1),(3,3),(4,5)\}$
Define $S \subseteq B \times C$ such that $S = \{(0,1),(0,2),(6,4),(6,1)\}$

A. Find R^{-1}.
B. Find $R \circ S$.
C. Find $S \circ R$.
D. Find S^{-1}.
E. Find $R^{-1} \circ S$.
F. Find $S^{-1} \circ R^{-1}$.
G. Find $(S \circ R)^{-1}$.
H. Find $R^{-1} \circ S^{-1}$.

For all exercises, subsequent, let $U = \mathbb{R}$ and $W = \mathbb{R} \times \mathbb{R}$.

Exercise 1.103. Let $D = \mathbb{N}_5$ and $E = \{-1, 0, 3, 5, 7, 11\}$.
Define $j \subseteq D \times E$ such that $k = \{(1,-1),(2,5),(3,0),(4,11),(5,7)\}$
Determine if the following statements are true or false; but, not both.
Justify why you opine that which you contend is the answer.

A. j is a relation from D to C

B. j is a function from D to C

C. j is an injection from D into C

D. j is a surjection from D onto C

E. j is a bijection from D onto C

F. j^{-1} is a relation from C to D

G. j^{-1} is a well defined function from C to D

Exercise 1.104. Let $D = \mathbb{N}_5$ and $C = \{-1, 0, 5, 7, 11\}$.
Define $k \subseteq D \times C$ such that $k = \{(1,-1),(2,0),(3,-1),(4,5),(5,7)\}$
Determine if the following statements are true or false; but, not both.
Justify why you opine that which you contend is the answer.

A. k is a relation from D to C

B. k is a function from D to C

C. k is an injection from D into C

D. k is a surjection from D onto C

E. k is a bijection from D onto C

F. k^{-1} is a relation from C to D

G. k^{-1} is a well defined function from C to D

Notation 1.105. Let $W = U \times V$ and $D \subseteq U$ whilst $C \subseteq V$. Suppose f is a well defined function from D to C. The notation $f : D \longrightarrow C$ is short-hand for what was just elucidated.[2]

Exercise 1.106. Consider $g : [0, \infty) \longrightarrow \mathbb{R}$ such that $g(x) = x^2$. Is it a bijection or is it not –justify why you opine such.

Exercise 1.107. Consider $h : \mathbb{R} \longrightarrow [0, \infty)$ such that $h(x) = x^2$. Is it a bijection or is it not –justify why you opine such.

Exercise 1.108. Consider $f_5 : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f_5(x) = x^2$. Is it a bijection or is it not –justify why you opine such.

Exercise 1.109. Consider $f_6 : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f_6(x) = x^3$. Is it a bijection or is it not –justify why you opine such.

Exercise 1.110. Consider $f_7 : [0, \infty) \longrightarrow [0, \infty)$ such that $f_7(x) = x^2$. Is it a bijection or is it not –justify why you opine such.

1.16. **Functions Part 2.** Let us consider a two-dimensional discussion of the concept of function. Please recall the **definition of a function**; it is very important to review it. A function is NOT:
(1) a rule;
(2) an input-output machine; or,
(3) an algebraic expression.
A **function** is a special kind of relation from a set to another set. The points are ordered-pairs (just as any point of a product set is a set of ordered pairs) and there is a restriction on what kind of relation is a function.

Definition 1.111. Let U be a well-defined universe and W be a well-defined universe. Let $V = U \times W$.
Let $D \subseteq U$ and $C \subseteq W$. Consider the relation, f, from D to C. f is a **well-defined function** from D to C *iff* $dom(f) \neq \varnothing$; $cor(f) = dom(f)$; and, $(x, p) \in f \wedge (x, q) \in f$ implies that $p = q$.

Notice the well-defined universe must be defined.

First, and foremost a <u>corange</u> and a <u>codomain</u> must be defined.

Then the <u>relation</u> is defined so that, of course means that a <u>domain</u> and a <u>range</u> exist.

[2]The problem with the notation is some use it and what they are claiming is not true (the thing is not a well defined function from D to C.

Then the function is defined either algebraically, point-by-point, or some other *rigourous* way. So, consider the following:

Exercises B Supplemental

Exercises B.9 - B.10 Let $U = V = W = \mathbb{R}$; $Y_1 = \mathbb{R} \times \mathbb{R}$; $Y_2 = \mathbb{R} \times \mathbb{N}$; and, $Y_3 = \mathbb{N} \times \mathbb{R}$. Let $A = \mathbb{N}_5$ and $B = \{-1, 0, 3, 5, 6\}$; and $C = \{1, 2, 4\}$.

Exercises B.9 Define $f \subseteq A \times B$ such that $f = \{(1, -1), (2, -1), (3, 3), (4, 5)\}$
1. Is f a well-defined function from A to B with respect to the universe Y_1?
2. Is f a well-defined function from A to B with respect to the universe Y_2?
3. Is f a well-defined function from A to B with respect to the universe Y_3?
Please note how important the discussion of the universe is from these questions.

Exercises B.10 A. Consider the universe Y_1 and note the relation g defined as $g \subseteq B \times C$ such that $g = \{(-1, 1), (3, 2), (6, 4), (6, 1)\}$
Is g a well-defined function from B to C?

B. Consider the universe Y_1 and note the relation h defined as $h \subseteq B \times C$ such that $h = \{(0, 1), (3, 2), (5, 4), (6, 1)\}$
Is h a well-defined function from B to C?

C. Consider the universe Y_1 and note the relation j defined as $j \subseteq B \times C$ such that $j = \{(-1, 2), (0, 1), (3, 2), (5, 4), (6, 1)\}$
Is j a well-defined function from B to C?

D. Consider the universe Y_1 and note the relation k defined as $k \subseteq B \times C$ such that $j = \{(-1, 2), (5, 4), (6, 1), (0, 1), (3, 2), (-1, 4)\}$
Is k a well-defined function from B to C?

Exercise 1.112. Let $U = \mathbb{R}$ and $V = \mathbb{R}$.
Let $Y_1 = U \times V$, Peter claims $f_1(x) = \sqrt{x}$ is a well defined function from U to V.
Is he correct or incorrect? Why?

Exercise 1.113. Let $U = \mathbb{N}$ and $V = \mathbb{N}$.
Let $Y_1 = U \times V$, Paula claims $f_2(x) = \sqrt{x}$ is a well defined function from U to V.
Is she correct or incorrect? Why?

Exercise 1.114. Let $W = \mathbb{R} \times \mathbb{R}$.
Let $U = \mathbb{N}$ and $V = \mathbb{R}$.
Let $Y_2 = U \times V$, Zach claims $f_3(x) = \sqrt{x}$ is a well defined function from U to V.
Is he correct or incorrect? Why?
What does $f_3(x)$ look like graphically in the plane ($W = \mathbb{R} \times \mathbb{R}$)?

Exercise 1.115. Let $W = \mathbb{R} \times \mathbb{R}$.
Let $U = \mathbb{N}$ and $V = \mathbb{R}$.
Let $Y_1 = U \times V$, Ana claims $f_4(x) = x^2$ is a well defined function from U to V.
Is she correct or incorrect? Why?
What does $f_4(x)$ look like graphically in the plane ($W = \mathbb{R} \times \mathbb{R}$)?

Exercise 1.116. Let $W = \mathbb{R} \times \mathbb{R}$.
Let $U = \mathbb{R}$ and $V = \mathbb{R}$.
Let $Y_1 = U \times V$, Betty claims $f_5(x) = x^2$ is a well defined function from U to V.
Is she correct or incorrect? Why? What does $f_4(x)$ look like graphically in the plane ($W = \mathbb{R} \times \mathbb{R}$)?

Exercise 1.117. Let $U = \mathbb{R}$ and $V = \mathbb{N}$.
Let $Y_1 = U \times V$, Carla claims $f_5(x) = x^2$ is a well defined function from U to V.
Is she correct or incorrect? Why?

Exercise 1.118. Let $U = \mathbb{N}$ and $V = \mathbb{N}$.
Let $Y_1 = U \times V$, Donald claims $f_5(x) = x^2 - 4$ **is not** a well defined function from U to V.
Is he correct or incorrect? Why?

Exercise 1.119. Let $W = \mathbb{R} \times \mathbb{R}$.
$U = \mathbb{N}$ and $V = \mathbb{R}$.
Let $f_6(x) = x^2 - 4$ Let $Y_1 = U \times V$, Donald claims $f_5(x) = x^2 - 4$ **is** a well defined function from U to V.
Is he correct or incorrect? Why?

2. Cardinality (Functions Part 3)

Definition 2.1. Let U be a well-defined universe and let A be a set.
The set A is **finite** means either:

(1) $A = \varnothing$; or,
(2) that there exists some well-defined bijection, f from A to an initial segment of the naturals, $\mathbb{N}_p$

A well-defined non-empty finite set is called a 'non-trivial' finite set.

Definition 2.2. Let U be a well-defined universe and let A be a set.
The cardinality of A is the number of distinct elements in A, and is denoted as $|A|$.

Comment 2.3. Let U be a well-defined universe and let $A = \varnothing$. It is the case that $|A| = 0$.

Comment 2.4. Let U be a well-defined universe and let $A \neq \emptyset$ be a non-trivial finite set and $|A| = p$, $p \in \mathbb{N}$. The value p is the size of the initial segment of naturals such that there is a well-defined bijection, f, such that $A \overset{f}{\rightarrowtail\!\!\!\!\twoheadrightarrow} \mathbb{N}_p$.

Notice the theory of finite sets and the axiom of null creates a richer and more sublime understanding of $\mathbb{N}^*$ as opposed to $\mathbb{N}$. We are now in a position to refer to $\mathbb{N}^*$ as the cardinal naturals which is how you may find them referred to and are designated as $\mathbb{N}_c$. Ergo, $\mathbb{N}^* = \mathbb{N}_c$

Definition 2.5. Let U be a well-defined universe and let A be a set.
The set A is **infinite** means it is not finite.

Definition 2.6. Let U be a well-defined universe and let A be a set.
The set A is **denumerable** means that there exists some well-defined bijection, f from A onto the naturals, $\mathbb{N}$.

Definition 2.7. Let U be a well-defined universe and let A and B be sets.
A set A that is finite or denumerable is a **countable** set.
A set B that is not finite nor denumerable is an **uncountable** set.

Sets can be finite, denumerable, or countable but not any of the two concurrently.

Exercise 2.8. Let $U = \mathbb{R}$ and let $A = \{\pi, \varphi, e, \sqrt{2}\}$.
Consider the relation $g \subseteq A \times \mathbb{N}_4$ such that $g = \{(\pi, 1), (\varphi, 2), (e, 3), (\sqrt{2}, 4)\}$. Note that g is a well defined function, it is injective, and surjective. Thus, it is used in a proof that $|A| = 4$. Verify g^{-1} is a well-defined bijection and $g^{-1} \subseteq \mathbb{N}_4 \times A$ also proves that $|A| = 4$ meaning that the definition of a non-empty finite set can be expressed as that there exists some well-defined bijection, f from an initial segment of the naturals, $\mathbb{N}_p$ ($p \in \mathbb{N}$), to A.

Exercise 2.9. Let $U = \mathbb{R}$ and let $A = \{\pi, \varphi, e, \sqrt{2}\}$.
Consider the relation $g \subseteq A \times \mathbb{N}_4$ such that $g = \{(\pi, 1), (\varphi, 2), (e, 3), (\sqrt{2}, 4)\}$. Note that g is a well defined function, it is injective, and surjective. Consider g^{-1} (which is a well-defined bijection and $g^{-1} \subseteq \mathbb{N}_4 \times A$).
1. Consider the relation $g \circ g^{-1}$. What is its corange, domain, codomain, and range?
2. Consider the relation $g^{-1} \circ g$. What is its corange, domain, codomain, and range?

Prove or disprove each claim.

Claim 2.10. Consider the relation $h \subseteq \mathbb{R} \times \mathbb{R}$ such that $h(x) = x^2$. h is a well defined bijection.

Claim 2.11. Consider the relation $g \subseteq \mathbb{R} \times \mathbb{R}$ such that $g(x) = x^3$. g is a well defined bijection.

2.1. More About Functions: Definitions and Theorems.

Definition 2.12. Let U be a well defined universe whilst A and B are sets. Let there be a well defined bijection, f, from A to B. With such a function existing between the sets, we say that A and B are **equinumerous**. We denote such as $|A| = |B|$ and also by the notation $A \sim B$.

Definition 2.13. Let U be a well defined universe and let A be a set. A is **infinite** if and only if A is not finite.

Definition 2.14. Let $U = \mathbb{R}$ and consider the set $\mathbb{N}$. We define the cardinality of $\mathbb{N}$ as $\aleph_0$. We denote such as $|\mathbb{N}| = \aleph_0$.

Theorem 2.15. Let U be a well defined universe whilst A and B are sets such that A and B are disjoint ($A \cap B = \emptyset$). $|A \cup B| = |A| + |B|$

Theorem 2.16. Let U be a well defined universe whilst A and B are sets such that $A \subseteq B$ It is the case that $|A| \leq |B|$

False Claim 2.17. Let U be a well defined universe whilst A and B are sets such that $A \subset B$ It is the case that $|A| < |B|$

Theorem 2.18. Let $U = \mathbb{R}$ and consider the set $\mathbb{N}^*$. $|\mathbb{N}^*| = \aleph_0$.

So,

Theorem 2.19. Let U be a well defined universe whilst A and B are sets such that $A \subset B$ It is the case that $|A| \leq |B|$

Let $U = \mathbb{R} \times \mathbb{R}$

Exercise 2.20. Consider the function $g : \mathbb{Z} \longrightarrow \mathbb{R}$ defined by:

$$g(x) = 3 \cdot x^2 + 1$$

A. Let $A = \{5, 13\}$ Find $g[A]$.
B. Let $B = \mathbb{N}_4$ Find $g[B]$.
C. Let $M = \{-2, -1, 1, 2, 3\}$ Find $g[M]$.
D. Why is it improper (wrong) to ask,"Let $C = [5, 13]$ Find $g[C]$?"

Exercise 2.21. Consider the function $f : \mathbb{N} \longrightarrow \mathbb{R}$ defined by:

$$f(x) = 3 \cdot x^2 + 1$$

A. Let $A = \{5, 13\}$ Find $f[A]$.
B. Let $B = \mathbb{N}_4$ Find $f[B]$.
C. Why is it improper (wrong) to ask,"Let $M = \{-2, -1, 1, 2, 3\}$ Find $f[M]$?"

Exercise 2.22. Consider the function $f : \mathbb{N} \longrightarrow \mathbb{R}$ defined by:

$$f(x) = 3 \cdot x^2 + 1$$

A. Let $A = \{5, 13\}$ Find $f^{-1}[A]$.
B. Let $B = [0, 50]$ Find $f^{-1}[B]$.
C. Let $C = [-5, 1]$ Find $f^{-1}[C]$.
D. Let $E = (-11, 0]$ Find $f^{-1}[E]$.

Exercise 2.23. Consider the function $f : \mathbb{N} \longrightarrow \mathbb{R}$ defined by:

$$f(x) = 3 \cdot x^2 + 1$$

A. Is f is injective? Explain.
B. Is f is surjective? Explain.
B. Is f is bijective? Explain.

Exercise 2.24. Consider the function $p : \mathbb{N}_6 \longrightarrow \mathbb{N}_6$ defined by:
$p = \{(1, 3), (2, 6), (3, 5), (4, 4), (5, 2), (6, 1)\}$ another way to write p is as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 4 & 2 & 1 \end{pmatrix}$$

such notation is called **permutation function notation** as discussed in class.

A. Find p^{-1}
B. Is p^{-1} a well defined function from $\mathbb{N}_6$ to $\mathbb{N}_6$?
C. Write p^{-1} in permutation function notation.

Exercise 2.25. Consider the permutation function $p : \mathbb{N}_6 \longrightarrow \mathbb{N}_6$ defined by:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 4 & 2 & 1 \end{pmatrix}$$

and the permutation function $q : \mathbb{N}_6 \longrightarrow \mathbb{N}_6$ defined by:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 5 & 6 & 1 \end{pmatrix}$$

A. Find $p \circ q$
B. Find $q \circ q$
C. Find $q \circ p$
D. Find $p \circ p$
E. Find $p^{-1} \circ q$
F. Find $q \circ q^{-1}$
G. Find $q^{-1} \circ p^{-1}$
H. Find $p^{-1} \circ p$

Exercise 2.26. Let $U = \mathbb{R} \times \mathbb{R}$

Consider the relation $d : \mathbb{Z} \longrightarrow \mathbb{N}$ defined by:

$$d(z) = \begin{cases} -2 \cdot z + 1, & z \leq 0 \\ \\ 2 \cdot z, & z > 0 \end{cases}$$

Is d a well defined function? Explain.

Exercise 2.27. Let $U = \mathbb{R} \times \mathbb{R}$
Assume the relation $d : \mathbb{Z} \longrightarrow \mathbb{N}$ defined by:

$$d(z) = \begin{cases} -2 \cdot z + 1, & z \leq 0 \\ \\ 2 \cdot z, & z > 0 \end{cases}$$

is a well defined function.
A. Prove or disprove d is injective.
B. Prove or disprove d is surjective.
C. Prove or disprove d is bijective.
D. From part A, B, and C what is the cardinality of $\mathbb{Z}$?

2.2. **Some Number Theory: Definitions and Theorems.**

Definition 2.28. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. A **common divisor** of p and q is a natural number, c, such that $p = c \cdot a \ni a \in \mathbb{N}$, and $q = c \cdot b \ni b \in \mathbb{N}$.

Theorem 2.29. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. Let the set G denote the set of common divisors of p and q. $G \neq \emptyset$.

Theorem 2.30. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. Let the set G denote the set of common divisors of p and q. $|G| < \aleph_0$.

Theorem 2.31. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. Let the set G denote the set of common divisors of p and q. $\langle G, \leq \rangle$ is a linear order.

Definition 2.32. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. The **greatest common divisor** (GCD) of p and q is the natural number, r such that $p = r \cdot a$, $a \in \mathbb{N}$, and $q = r \cdot b$ $b \in \mathbb{N}$ and no other natural number, w is a common divisor of p and q where $w > r$.

Theorem 2.33. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. Let r be the greatest common divisor of p and q and let the set G be the set of common divisors of p and q. r is the maximal element in $\langle G, \leq \rangle$ (the linear order of G under the usual order (less than or equal to inherited from the axioms of the reals)). Since G is finite and all elements of G are comparable r is greatest element of G by definition.

Definition 2.34. Let $U = \mathbb{R}$ and consider the set $A = \mathbb{N}$. Let p and q be elements of $\mathbb{N}$. p and q are **relatively prime** iff the GCD of p and q is 1. Further, $G = \{1\}$ where G is the set of common divisors of p and q.

Theorem 2.35. Let $U = \mathbb{R}$ and consider the set $B = \mathbb{Q}$. Let $p \in \mathbb{Q}$ where $p > 0$.
$\exists\, m \in \mathbb{N} \wedge n \in \mathbb{N} \ni p = \dfrac{m}{n}$ and m and n are relatively prime.

Definition 2.36. Let $U = \mathbb{R}$ and consider the set $B = \mathbb{Q}$. Let $p \in \mathbb{Q}$ where $p > 0$. Let $p = \dfrac{m}{n}$ where m and n are relatively prime and are in the set $\mathbb{N}$. The number p is said to be in **reduced form**.

Theorem 2.37. Let $U = \mathbb{R}$ and consider the set $C = \mathbb{I}$. $\quad C \neq \emptyset$.

. .

2.3. **Larger Cardinals than aleph-null and a Peek Ahead.** . . Recall, let $U = \mathbb{R}$.

$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}. \quad \mathbb{Q} \cap \mathbb{I} = \emptyset.$
We claim the following:

Theorem 2.38. Let $U = \mathbb{R}$ and consider $\mathbb{Q}$. $|\mathbb{Q}| = \aleph_0$.

Theorem 2.39. Let $U = \mathbb{R}$ and consider $\mathbb{I}$. $|\mathbb{I}| > \aleph_0$.

During this semester (Real Analysis I) you will prove some interesting results, I hope, such as:

Theorem 2.40. Let $U = \mathbb{R}$ and let a and b be distinct points in $\mathbb{Q}$.
$\exists\, c \in \mathbb{Q} \ni c$ is between a and b.

Theorem 2.41. Let $U = \mathbb{R}$ and let a and b be distinct points in $\mathbb{Q}$.
$\exists\, k \in \mathbb{I} \ni k$ is between a and b.

Theorem 2.42. Let $U = \mathbb{R}$ and let s and t be distinct points in $\mathbb{I}$.
$\exists\, c \in \mathbb{Q} \ni c$ is between s and t.

Theorem 2.43. Let $U = \mathbb{R}$ and let s and t be distinct points in $\mathbb{I}$.
$\exists\, w \in \mathbb{I} \ni w$ is between s and t.

Theorem 2.44. Let $U = \mathbb{R}$ and let a be a point in $\mathbb{Q}$ and t be a point in $\mathbb{I}$.
$\exists\, w \in \mathbb{I} \ni w$ is between a and t.

Theorem 2.45. Let $U = \mathbb{R}$ and let a be a point in $\mathbb{Q}$ and t be a point in $\mathbb{I}$.
$\exists\, c \in \mathbb{Q} \ni c$ is between a and t.

Theorem 2.46. Let $U = \mathbb{R}$ and let x and y be distinct points in $\mathbb{R}$.
$\exists\, p \in \mathbb{Q} \ni p$ is between x and y.

Theorem 2.47. Let $U = \mathbb{R}$ and let x and y be distinct points in $\mathbb{R}$.
$\exists\, w \in \mathbb{I} \ni w$ is between x and y.

Theorem 2.48. Let $U = \mathbb{R}$ and let (a, b) be a segment.
$\exists$ infinitely many points $w \in \mathbb{I} \ni w \in (a, b)$.

Theorem 2.49. Let $U = \mathbb{R}$ and let (a, b) be a segment.
$\exists$ infinitely many points $p \in \mathbb{Q} \ni p \in (a, b)$.

Fascinating.