# Calculus III

## Handy Dandy Guide to Series Part III

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A **power series at $\mathbf{x} = \mathbf{c}$** is a series such that it is of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n \qquad \forall \quad n \in \mathbb{N}^* \quad a_n \in \mathbb{R}$$

.

Let $D \subseteq \mathbb{R} \quad \wedge \quad [a,b] \subseteq D$.
Let $f : D \longrightarrow \mathbb{R}$ be a well defined function such that $f^{(n)}(x)$ exists $\quad \forall \quad x \in (a,b)$.
Let $\quad c \in (a,b)$.

The **Taylor Series for $f(\mathbf{x})$ at $\mathbf{x} = \mathbf{c}$** is a power series such that we will denote it as $T(x)$ where

$$f(x) = T(x) = f(c) + \sum_{n=1}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} \qquad \forall x \in K$$

where $K$ is the set of all points in the interval of convergence for $T(x)$. So, obviously $c \in K$.

The **Maclaurin series** is the case $c = 0$ for a Taylor series of the function $f(x)$.
There are some Maclaurin series of import that we need to recall or produce quickly for applications and approximations, which are as follows:

(1)

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}$$

(2)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

(3)

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad \forall x \in [-1,1]$$

(4)

$$\ln(1+x) = \sum_{n=}^{\infty} \frac{(-1)^{(n+1)} x^n}{n} \quad \forall x \in (-1,1]$$

(5)

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

(6)

$$\frac{1}{1-x} = 1 + \sum_{n=1}^{\infty} x^n \quad \forall x \in (-1,1)$$

If one is smart then they DO NOT have to memorise these others for they are easily derived:

(1)

$$\cos x = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}$$

(2)

$$\cosh x = 1 + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

(3)

$$\frac{1}{1+x^2} = 1 + \sum_{n=1}^{\infty} (-1)^n \cdot x^{2n} \quad \forall x \in (-1,1)$$

This begs the question, why are they easily derived?
Well, for example consider

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad \forall x \in [-1,1]$$

$$f(x) = \arctan(x) \Rightarrow f^{(1)}(x) = \frac{1}{1+x^2}$$

Hence, recall from Calculus I, taking the derivative with respect to $x$ for the Maclaurin series for $\arctan x$ will help us determine the Maclaurin series for $\frac{1}{1+x^2}$.

Now, let $x \in (-1,1)$
Since $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad \forall x \in [-1,1]$,

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} + \dots \quad \Rightarrow$$

$$\frac{d}{dx}(\arctan(x)) = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} + \dots \quad \Rightarrow$$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \quad \Rightarrow$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} + \dots \quad \Rightarrow$$

$$\frac{1}{1+x^2} = 1 + \sum_{n=1}^{\infty} (-1)^n \cdot x^{2n} \quad \ni x \in (-1,1)$$

What is key is noting the countably infinite generalisation of the sum rule for derivatives.

Such is also the case for integrals!
So, recall from Calculus II

$$\int (\sinh(x))\,dx = \cosh(x) + k \qquad \ni k \in \mathbb{R}$$

$$\int (\sinh(x))\,dx = \int \left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right) dx$$

$$\int (\sinh(x))\,dx = \int \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \dots\right) dx$$

$$= \int (x)\,dx + \int \left(\frac{x^3}{3!}\right) dx + \int \left(\frac{x^5}{5!}\right) dx + \int \left(\frac{x^7}{7!}\right) dx + \int \left(\frac{x^9}{9!}\right) dx + \int \left(\frac{x^{11}}{11!}\right) dx + \int \left(\frac{x^{13}}{13!}\right) dx + \dots$$

$$= C + \left(\frac{x^2}{2!}\right) + \left(\frac{x^4}{4!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^8}{8!}\right) + \left(\frac{x^{10}}{10!}\right) + \left(\frac{x^{12}}{12!}\right) + \left(\frac{x^{14}}{14!}\right) + \dots \ni C \in \mathbb{R}$$

Let $C = k + 1$

$$\int (\sinh(x))\,dx = \int \left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right) dx$$

$$= k + 1 + \left(\frac{x^2}{2!}\right) + \left(\frac{x^4}{4!}\right) + \left(\frac{x^6}{6!}\right) + \left(\frac{x^8}{8!}\right) + \left(\frac{x^{10}}{10!}\right) + \left(\frac{x^{12}}{12!}\right) + \left(\frac{x^{14}}{14!}\right) + \dots \ni k \in \mathbb{R}$$

$$\Rightarrow$$

$$\int (\sinh(x))\,dx = k + 1 + \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \ni k \in \mathbb{R}$$

$$\int (\sinh(x))\,dx = k + \cosh(x) \ni k \in \mathbb{R}$$

Supercalifragilisticexpialidocious!