MATH 255 SET THEORY - DR. MCLOUGHLIN'S CLASS
HANDOUT 13
PAUSE TO CONSIDER WHERE WE ARE AND FROM WHENCE WE CAME

We started this section of the course building from sets to the more general product sets.

Recall:
Let U be a well defined universe. Let V be the well defined universe, $\mathcal{P}(U)$.

Let A be a set. We considered theorems on sets, power sets, $\mathcal{P}(\mathrm{A})$, and sub-collections of power sets, Ω.

Then we considered Let U be a well defined universe. Let V be the well defined universe..
Let A be a set within U. Let B be a set within V.
We defined the universe $W = U \times V$.

Def. 13.1 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $\mathrm{A} \subseteq U$ whilst $\mathrm{B} \subseteq V$. The relation M from A to B is the set
$\mathrm{M} = \{ (a, b): a \in \mathrm{A}, b \in \mathrm{B}\}$.

Let R be any subset of $\mathrm{A} \times \mathrm{B}$:

$\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$

R is called a relation . Big whoop.!

We found the subject of relations in and by themselves was not too intriguing.

So, we considered relations on a set.

Def. 13.2 : Let the universe $W = U \times U$ be defined from the well defined universe U such that $\mathrm{A} \subseteq U$. The relation S from A to A (S is on A) is the set
$\mathrm{S} = \{ (a, b): a \in \mathrm{A}, b \in \mathrm{A}\}$.

This creates more interesting ideas.

Def. 13.3 : Let the universe $W = U \times U$ be defined from the well defined universe U such that $\mathrm{A} \subseteq U$. Consider the relation R on A.
R is said to be reflexive if for every $x \in \mathrm{A}$, $(x, x) \in \mathrm{R}$ (note there must exist an x such that $(x, x) \in \mathrm{R}$, this is to ensure that $\varnothing$ does not vacuously hold as reflexive).

Def. 13.4 : Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A.
R is said to be *symmetric* if for every (x. y) $\in$ R $\Rightarrow$ (y, x) $\in$ R.

Def. 13.5 : Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A.
R is said to be *antisymmetric* if (x. y) $\in$ R $\Rightarrow$ (y, x) $\notin$ R .

Alternate Def. 13.5.1 : Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A.
R is said to be *symmetric* if (x. y) $\in$ R $\wedge$ (y, x) $\in$ R $\Rightarrow$ x = y.

Exercise:

1. Prove definition 13.5 and alternate definition 13.5.1 are logically equivalent.

Def. 13.6 : Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Consider the relation R on A.
R is said to be *transitive* if (x. y) $\in$ R $\wedge$ (y, z) $\in$ R $\Rightarrow$ (x, z) $\in$ R

Notice this is *not* the same as saying the following:
Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$.
Consider the relation R on A.
$(x, y) \in R \quad \Rightarrow \quad \exists \; z \ni \; (x, z) \in R \quad \wedge \quad (z, y) \in R$

Examples:

1. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation R on A.

Consider A = {1, 2, 3, 4} Let R = {(1, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)}

Exercises:

2. Prove R is not reflexive.
3. Prove R is not symmetric.
4. Prove R is not antisymmetric.
5. Prove R is not transitive.

2. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A.

Consider A = {1, 2, 3, 4} Let S = {(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)}

Exercises:

6. Prove S is reflexive.
7. Prove S is symmetric.
8. Prove S is not antisymmetric.
9. Prove S is transitive.

2. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation T on A.

Consider A = {1, 2, 3, 4} Let T = {(1, 1), (1, 2), (1, 3), (2, 3), (2, 2), (1, 4), (3, 3), (4, 4)}

Exercises:

10. Prove T is reflexive.
11. Prove T is not symmetric.
12. Prove T is antisymmetric.
13. Prove T is transitive.

Def. 13.7: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let the relation be R on the set A. If R is reflexive, symmetric, and transitive, then R is called an equivalence relation on A.

Example:

Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A.

Consider A = {1, 2, 3, 4} Let S = {(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)}

S is an equivalence relation on A.

Def. 13.8: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let the relation be R on the set A. If R is an equivalence relation on A, then R induces a subset of $\mathcal{P}(A)$ [a collection], called the collection of equivalence classes of A induced by R.
An equivalence class of the element $a \in A$ from R is denoted as $[a]_R$
$[a]_R = \{ b \in A: (a, b) \in R\}$

Example:

Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A.

Consider A = {1, 2, 3, 4} Let S = {(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)}

$[1]_S = \{1\}$. $[2]_S = \{2, 3\} = [3]_S$. $[4]_S = \{4\}$. Notice that $\Psi = \{[1]_S, [2]_S, [4]_S\} =$
$\{\{1\}, \{2, 3\}, \{4\}\} \subseteq \mathcal{P}(A)$.

Def. 13.9: Let the universe U be defined and let the set A be well defined such that $A \subseteq U$. Let the collection $\Pi \subseteq \mathcal{P}(A)$. Π is called a partition of A iff:

(1) $\forall\ M \in \Pi \wedge \forall\ N \in \Pi$, it is the case that $M \cap N = \varnothing$ (the sets in Π are pair-wise disjoint or they are said to be mutually exclusive); and
(2) $\bigcup\Pi = A$ (the collection is said to be exhaustive).

Theorem 13.1: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let the R be an equivalence relation on the set A. The collection of equivalence classes of A induced by R partitions A.

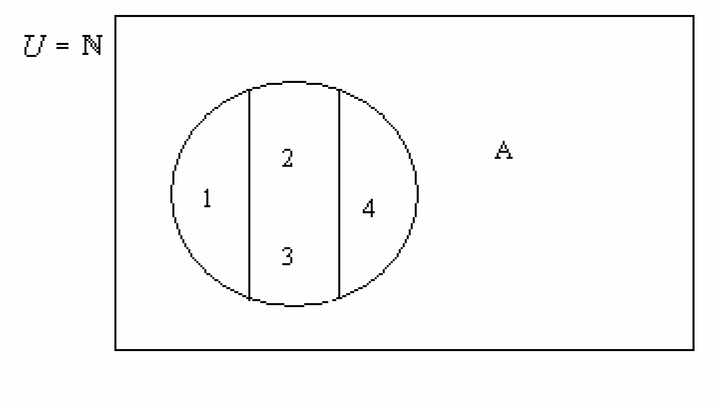
Theorem 13.2: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let Π be a partition of A. Π induces an equivalence relation on A which is typically denoted as $A\backslash\Pi$.

Example:

Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation S on A.

Consider A = {1, 2, 3, 4} Let S = {(1, 1), (2, 2), (3, 2), (2, 3), (3, 3), (4, 4)}

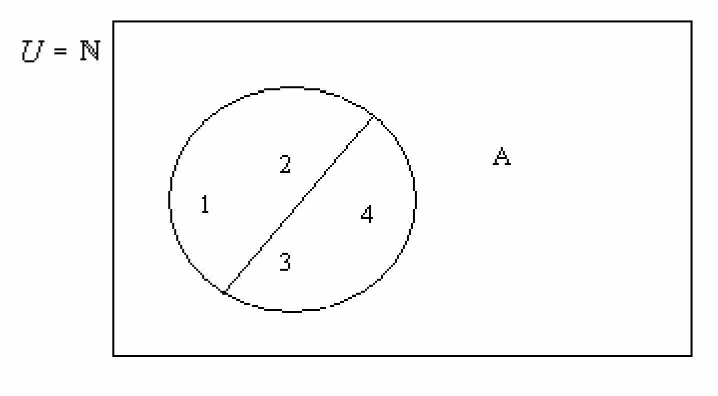
$[1]_S = \{1\}$. $[2]_S = \{2, 3\} = [3]_S$. $[4]_S = \{4\}$. Notice that $\Psi = \{[1]_S, [2]_S, [4]_S\}$ partitions A.



Venn Diagramme of Partition induced by Ψ

Example:

Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the partition of A, $\Phi = \{\{1, 2\}, \{3, 4\}\}$.



Venn Diagramme of Partition defined by Φ

Note that A\ Φ is an equivalence relation on A such that

A\ Φ = {(1, 1), (2, 2), (1, 2), (2, 1), (3, 3), (4, 4), (3, 4), (4, 3)}.

Def. 13.10: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let the relation be R on the set A. If R is reflexive, antisymmetric, and transitive, then R is called a partial ordering on A.

Def. 13.11: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be partially ordered under R. If either xRy or yRx for $x, y \in A$ then x and y are said to be comparable.

Def. 13.12: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under R. Let the symbol $\preceq$ denote a partial ordering.

Def. 13.13: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. If for every x and y in A x $\preceq$ y or y $\preceq$ x, then A is linearly ordered by $\preceq$.

Def. 13.14: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. If there exists an element l of A such that for every x in A $l \preceq$ x, then l is called the least element of A.

Def. 13.15: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. Let m belong to A. If there does not exist an element x of A such that x $\preceq$ m, then m is called a minimal element of A.

Def. 13.16: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. If there exists an element L of A such that for every x in A x $\preceq L$, then L is called the greatest element of A.

Def. 13.17: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. Let M belong to A. If there does not exist an element x of A such that M $\preceq$ x, then M is called a maximal element of A.

Def. 13.18: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. If a $\preceq$ b and $a \neq b$, then we will say a is less than b under the order and denote this by $a \prec b$.

Def. 13.19: Let A be a set and it be partially ordered under $\preceq$. If a $\preceq$ b and $a \neq b$, then we will say b is greater than a under the order and denote this by $a \prec b$.

Def. 13.20: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$ and $B \subseteq A$. Let β belong to A. If $x \preceq \beta \quad \forall \quad x \in B$ then β is called an upper bound of B.

Def. 13.21: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$ and $B \subseteq A$. Let b belong to A. If $b \preceq x \quad \forall \quad x \in B$ then b is called a lower bound of B.

Def. 13.22: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$ and $B \subseteq A$. Let δ belong to A. If $\delta \preceq \beta \; \forall \; \beta$ that are upper bounds of B then δ is called the least upper bound of B (or supremum) and is denoted as δ = lub(B).

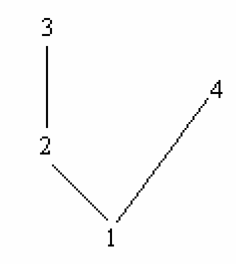
Def. 13.23: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$ and $B \subseteq A$. Let g belong to A. If $b \preceq g \; \forall \; b$ that are lower bounds of B then g is called the greatest lower bound (or infinum) of B and is denoted as g = glb(B).

Example:

Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation T on A.

Consider A = {1, 2, 3, 4} Let T = {(1, 1), (1, 2), (1, 3), (2, 3), (2, 2), (1, 4), (3, 3), (4, 4)}

Note that T is a partial order on A.



Hasse Diagramme of the partial order T on A.

It can be seen that 1 is a minimal element, 1 is the least element, 3 is a maximal element, 4 is a maximal element, and there does not exist a greatest element.

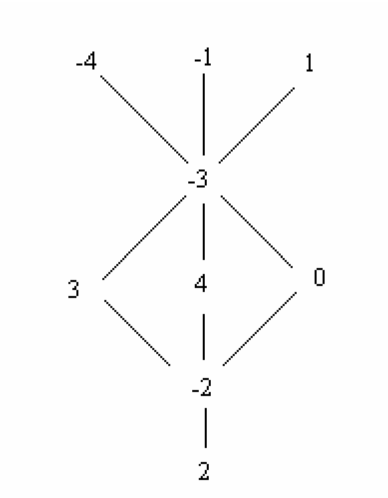
Consider $B \subseteq A \ni B = \{2, 4\}$
It can be seen that 1 is a lower bound of B and there is no upper bound.

Consider $C \subseteq A \ni C = \{2, 3\}$
It can be seen that 1 is a lower bound, 2 is a lower bound of B, 3 is an upper bound, and that 1 is not the glb(C), 2 is the glb(C), and 3 is the lub(C).

Example:

Let the universe $W = \mathbb{Z} \times \mathbb{Z}$. Let $K \subseteq \mathbb{Z} \ni K = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

Consider this Hasse diagramme:



It can be deduced that the Hasse diagramme is a graphical representation of a partial order on K, let us symbolise it as $\sqsupseteq$ and call the partial order $\mathfrak{S}$ on K.

So, $2 \sqsupseteq 2$, $2 \sqsupseteq -2$, $2 \sqsupseteq 3$, $-2 \sqsupseteq 3$, $-2, \sqsupseteq 4$, $-2 \sqsupseteq 0$, yada, yada, yada.

Notice that $4 \not\sqsupseteq 0$ and $0 \not\sqsupseteq 4$.

It can be seen that 2 is a minimal element, 2 is the least element, -4 is a maximal element, -1 is a maximal element, 1 is a maximal element, and there does not exist a greatest element.

Exercise:

14. Show that $\mathfrak{S}$ is indeed a partial order on K.

15 - 19 Find upper bounds of the sets relative to K under $\sqsupseteq$ (the partial order $\mathfrak{S}$ on K).

15. Consider $B \subseteq K \ni B = \{2, 4\}$
16. Consider $C \subseteq K \ni C = \{2, 3\}$
17. Consider $D \subseteq K \ni D = \{-2, 4, -3, -1\}$
18. Consider $E \subseteq K \ni E = \{-2, 3, -3, -4\}$
19. Consider $F \subseteq K \ni F = \{3, 0, -1\}$

We could study partial orders in even more detail. If one is so inclined, it is a great subject for directed reading and can lead to a nifty Senior Seminar project.

<u>Def. 13.24</u>: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$ and $B \subseteq A$. B is called a chain if all the elements of B are related, that is $\forall$ x $\forall$ y in B $x \preceq y$ or $y \preceq x$.

<u>Def. 13.25</u>: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$ and A be a chain.
Then, A is called a totally ordered set.

<u>Def. 13.26</u>: Let the universe $W = U \times U$ be defined from the well defined universe U such that $A \subseteq U$. Let A be a set and it be partially ordered under $\preceq$. A is called a well ordered set if for each non-empty subset of A, $B \subseteq A$ B has a "first element."
(e.g.: $\exists\ f \in B \ni f \preceq x\ \forall\ x \in B$)

<u>Def. 13.27</u>: An algebra is an ordered pair (A, *) such that A is a non-empty set and * is a binary operation on A.

<u>Def. 13.28</u>: Let $(A, *_A)$ and $(B, *_B)$ be algebras. $f\colon A \longrightarrow B$ is a homomorphism from A to B if $\forall x, y \in A \quad f(x *_A y) = f(x) *_B f(y)$

Def. 13.28 can be generalised to more operations (not necessarily binary) with distinguished elements for each operation (like the integers with addition and multiplication coordinated with distinguished elements zero and one, respectively)

<u>Def. 13.29</u>: Let $(A, +_A, *_A, 0_A, 1_A)$ and $(B, +_B, *_B, 0_B, 1_B)$ be algebras. $f\colon A \longrightarrow B$ is a homomorphism from $(A, +_A, *_A, 0_A, 1_A)$ to $(B, +_B, *_B, 0_B, 1_B)$ if:

$\forall x, y \in A \quad f(x *_A y) = f(x) *_B f(y)$

$\forall x, y \in A \quad f(x +_A y) = f(x) +_B f(y)$

$f(0_A) = f(0_B)$ and $f(1_A) = f(1_B)$

<u>Def. 13.30</u>: Let $(A, +_A, *_A, 0_A, 1_A)$ and $(B, +_B, *_B, 0_B, 1_B)$ be algebras and $f\colon A \longrightarrow B$ is a bijection from A to B that is a homomorphism from $(A, +_A, *_A, 0_A, 1_A)$ to $(B, +_B, *_B, 0_B, 1_B)$, then we call the homomorphism an isomorphism. Indeed, we say $(A, +_A, *_A, 0_A, 1_A)$ is isomorphic to $(B, +_B, *_B, 0_B, 1_B)$.

Define the relation "$\approx$" on A to be $x \approx y$ iff $f(x) = f(y)$.

i) $F\colon A \longrightarrow (A/\approx)$ defined by $F(x) = [x]$ is onto $A/\approx$

ii) if f is onto also, then there exists a unique function $\rho\colon (A/\approx) \longrightarrow B$ such that ρ is a bijection and $\rho \circ F = f$.

Last revised 8 April 2004