MATH 255 SET THEORY - DR. MCLOUGHLIN'S CLASS
HANDOUT 16
FUNCTIONS

Recall:
Let U be a well defined universe. Let V be the well defined universe, $\mathcal{P}(U)$.

Let A be a set. We considered theorems on sets, power sets, $\mathcal{P}$(A), and sub-collections of power sets, Ω.

Then we considered Let U be a well defined universe. Let V be the well defined universe..
Let A be a set within U. Let B be a set within V.
We defined the universe $W = U \times V$.

<u>Def. 13.1</u> : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The relation M from A to B is the set
M = { (a, b): a ∈ A, b ∈ B}.

Let R be any subset of $A \times B$, $R \subseteq A \times B$, R is called a relation . Big whoop.!
Please remember for relations
dom(R) = A, cod(R) = B, and the range, ran(R) ⊆ B such that
ran(R) = {x ∈ B: ∃ a ∈ A ∍ (a, x) ∈ R}.

<u>Def. 16.1</u> : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The relation f from A to B is the set
f = { (a, b): a ∈ A, b ∈ B}. Let the relation f have the following properties:

(1) ∀ a ∈ A ∃ b ∈ B ∍ (a, b) ∈ f ; and,

(2) (a, x) ∈ f ∧ (a, y) ∈ f ⇒ x = y,

then it is the case that f is called a function from A to B (or more precisely a well-defined function from A to B) and we symbolise it as f: A ⟶B or $A \xrightarrow{f} B$

<u>Def. 16.2</u> : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. A function $f: A \to B$ is injective (or one-to-one) iff for every x and for every y in A, $f(x) = f(y) \Rightarrow x = y$.

Notation: Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The function $f: A \to B$ which is injective is denoted as f: A $\rightarrowtail$ B

Def. 16.3 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. A function $f : A \to B$ is surjective (or onto B) iff for every y in B there exists an x in A such that $f(x) = y$

Notation: Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The function $f : A \to B$ which is surjective is denoted as f: A $\twoheadrightarrow$ B

Def. 16.4 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. A function $f : A \to B$ is bijective iff f is injective and surjective.

Notation: Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. The function $f : A \to B$ which is bijective is denoted as f: A $\rightarrowtail\!\!\!\twoheadrightarrow$ B

Def. 16.5 : Let the universe $W = U \times \mathbb{R}$ be defined such that $A \subseteq U$ whilst $B \subseteq \mathbb{R}$. The f is called a function from A to B (or more precisely a well-defined function from A to B) is called a real-valued function (since the cod(f) $\subseteq \mathbb{R}$ [one can just as easily let the codomain be the reals]).

Def. 16.6 : Let the universe $W = \mathbb{N} \times \mathbb{R}$ be defined such that $A \subseteq \mathbb{N}$ whilst $B \subseteq \mathbb{R}$. The f is called a sequence from A to B (or more precisely a well-defined function from A to B) since it is a real-valued function (since the dom(f) $\subseteq \mathbb{N}$ and the cod(f) $\subseteq \mathbb{R}$).

Examples:

1. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation f on A.

Consider A = {1, 2, 3, 4} Let f = {(1, 1), (1, 2), (2, 2), (3, 2), (4, 3)}

Exercises:
1. Prove f is not a function from A to A.

2. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation g on A.

Consider A = {1, 2, 3, 4} Let f = {(1, 1), (2, 2), (3, 2), (4, 4)}

Exercises:

2. Prove g is a function from A to A.
3. Find ran(g).

3. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation h on A.

Consider A = $\mathbb{N}_4$ Let h = {(1, 1), (2, 3), (3, 4), (4, 2)}

4. Prove h is a function from A to A.
5. Find ran(h).

4. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation f_1 from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_4$ Let f_1 = {(1, 6), (2, 0), (3, -1), (4, π)}

6. Prove f_1 is a real-valued function.
7. Find ran(f_1).

5. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation f_2 from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_{25}$ Let $f_1 : A \longrightarrow \mathbb{R} \ni f_1(a) = \sqrt{a}$

7. Prove f_2 is a real-valued function.
8. Find ran(f_2).

6. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Consider the relation f_3 from $\mathbb{N}$ to $\mathbb{R}$.

Let $f_3 : \mathbb{N} \longrightarrow \mathbb{R} \ni f_3(a) = 3a + 1$.

9. Prove f_3 is a sequence.
10. Find ran(f_2).

Def. 16.7 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $f: A \longrightarrow B$ be a function from A to B. Let $C \subseteq A$. Define $f|_C$ as the function $f|_C : C \longrightarrow B \ni f|_C(c) = f(c)$ defined by $f: A \longrightarrow B$ $f|_C$ is called the restriction function of f on C.

Def. 16.8 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $f: A \longrightarrow B$ be a function from A to B. Let $A \subseteq M \subseteq U$. Define $f|^M$ as a function $f|^M : M \longrightarrow B \ni f|^M(m) = f(m)$ defined by $f: A \longrightarrow B$ and such that $\forall$ x $\in$ (M – A) $f|^M$ is defined properly (who cares how just properly) so that $f|^M$ is a function from M to B. $f|^M$ is called an extension function of f on M.

Example:

1. Let the universe $W = \mathbb{Z} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation f_5 from A to $\mathbb{R}$.

Consider A = $\mathbb{N}_{25}$ Let $f_5 : A \longrightarrow \mathbb{R} \ni f_5(a) = \sqrt{a}$

Let C = $\mathbb{N}_{10}$. Let M = $\mathbb{N}_{28}$. Let L = {1, 4, 7, 10}. Let J = {-4, -3, -2, -1, 0, 1, 2, 3, 4}

11. Find $f_5|_C$ (note that it is unique).
12. Find a $f_5|^M$ (note that it is not unique).
13. Find $f_5|_L$ (note that it is unique).
14. Try to create $f_5|^J$ - - note it cannot be defined.

<u>Def. 16.9</u> : Let the universe $W = U \times V \times X$ be defined from the well defined universes U, V, X such that $A \subseteq U$, $B \subseteq V$, whilst $C \subseteq X$. Let f be a function from A to B and g be a function from B to C. Define the composition function[1] $g \circ f$: A $\longrightarrow$C such that
$g \circ f(x) = g(f(x))$.

So, since f: A $\longrightarrow$ B is a well defined function $\exists$ y $\in$ B $\ni f(x) = y$. Since g: B $\longrightarrow$ C is a well defined function, dom(g) = B. Hence, $\exists$ z $\in$ C $\ni g(y) = z$.

So, we have $A \xrightarrow{f} B \wedge B \xrightarrow{g} C \Rightarrow A \xrightarrow{g \circ f} C$

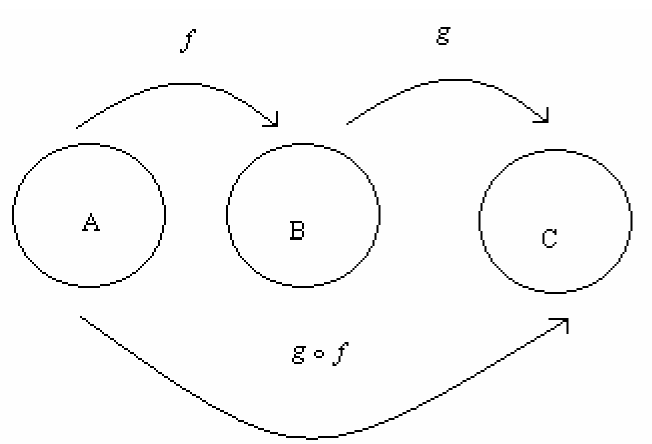

Diagramme depicting $A \xrightarrow{f} B \wedge B \xrightarrow{g} C \Rightarrow A \xrightarrow{g \circ f} C$

[1] Same definition as composition relation but just subsumed to functions.

Theorem 16.1 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let the function $f: A \to B$ be bijective. It is the case that the relation $f^{-1} \subseteq B \times A$ is also a bijective function and note that $f^{-1}: B \to A$.
So, as $f: A \rightarrowtail\!\!\!\twoheadrightarrow B \Rightarrow f^{-1}: B \rightarrowtail\!\!\!\twoheadrightarrow A$.

Def. 16.10 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $C \subseteq A$. Let f be a function from A to B.
The set $f[C]$ is a subset of B and is called the image set of C under f
$f[C] = \{d \in B: f(c) = d \text{ where } c \in C\}$.

Def. 16.11 : Let the universe $W = U \times V$ be defined from the well defined universes U and V such that $A \subseteq U$ whilst $B \subseteq V$. Let $M \subseteq B$. Let f be a function from A to B.
The set $f^{-1}[M]$ is a subset of A and is called the inverse image set of M under f
$f^{-1}[M] = \{a \in A: f(a) = m \text{ where } m \in M\}$.

Note: The existence of the set $f^{-1}[M]$ does not imply that f^{-1} is a function from B to A. We know that since f is a function from A to B it was a relation from A to B; hence, f^{-1} is a relation from B to A but is not necessarily a function from B to A.

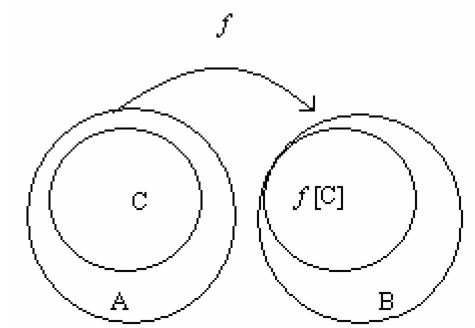

Diagramme depicting image set $f[C]$

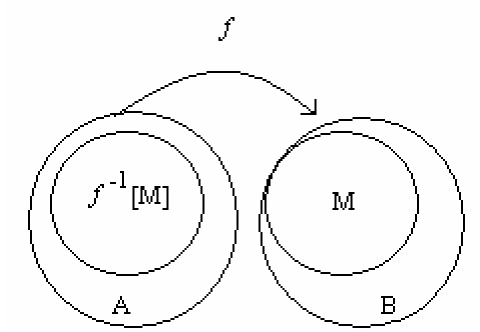

Diagramme depicting inverse image set $f^{-1}[M]$

Example:

1. Let the universe $W = \mathbb{Z} \times \mathbb{R}$. Let $A \subseteq \mathbb{Z}$. Consider the function f_6 from A to $\mathbb{R}$.

Consider A = { -2, -1, 0, 1, 2} Let $f_6 : A \longrightarrow \mathbb{R} \ni f_6(a) = a^2$

Let C = {-1, 0, 1, 2} Notice that $f[C] = \{d \in B: f(c) = d$ where $c \in C\}$ so
$f[C] = \{0, 1, 4\}$.

Let $M = [\frac{1}{\sqrt{2}}, 9)$. Notice that $f^{-1}[M] = \{a \in A: f(a) = m$ where $m \in M\}$ so
$f^{-1}[M] = \{-1, 1, 2\}$.
Let K = [¼ , ¾). Notice that $f^{-1}[K] = \varnothing$.
Let C = {-1, 0, 1, 2} Notice that $f^{-1}[C] = \{-1, 0, 1\}$.
Let L = (-∞, 0) Notice that $f^{-1}[L] = \varnothing$.

2. Let the universe $W = \mathbb{N} \times \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Consider the relation *g* on A.

Consider A = {1, 2, 3, 4} Let *f* = {(1, 1), (2, 2), (3, 2), (4, 4)}

Exercises:

15. Let C = {2, 3}. Find the set *f*[C].
16. Let M = {3, 4}. Find the set $f^{-1}[M]$.
17. Let K = {3, 4}. Find the set *f*[K].
18. Let L = {2, 3}. Find the set $f^{-1}[L]$.

3. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation *h* on A.

Consider $A = \mathbb{N}_4$ Let *h* = {(1, 1), (2, 3), (3, 4), (4, 2)}

19. Let C = {2, 3}. Find the set *h*[C].
20. Let M = {3, 4}. Find the set $h^{-1}[M]$.
21. Let K = {3, 4}. Find the set *h*[K].
22. Let L = {2, 3}. Find the set $h^{-1}[L]$.

5. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Let $A \subseteq \mathbb{N}$. Consider the relation f_2 from A to $\mathbb{R}$.

Consider $A = \mathbb{N}_{25}$ Let $f_1 : A \longrightarrow \mathbb{R} \ni f_1(a) = \sqrt{a}$

23. Let C = {2, 3}. Find the set $f_1[C]$.
24. Let M = {3, 4}. Find the set $f_1^{-1}[M]$.

25. Let K = {3, 4}. Find the set f_1[K].
26. Let L = {2, 3}. Find the set f_1^{-1}[L].

6. Let the universe $W = \mathbb{N} \times \mathbb{R}$. Consider the relation f_3 from $\mathbb{N}$ to $\mathbb{R}$.

Let $f_3 : \mathbb{N} \longrightarrow \mathbb{R} \ni f_3(a) = 3a + 1$.

27. Let J = $\mathbb{N}_4$. Find the set f_3[J].

28. Let M = $\mathbb{N}_4$. Find the set f_3^{-1}[M].

29. Let K = {3, 8}. Find the set f_3[K].
30. Let L = [3, 8]. Find the set f_3^{-1}[L].

We could study functions in even more detail. If one is so inclined, it is a great subject for directed reading and can lead to a nifty Senior Seminar project.

Definition Set: Assume the universe $W = \mathbb{R} \times \mathbb{R}$ be defined.

Let that $A \subseteq \mathbb{R}$ whilst $B \subseteq \mathbb{R}$.

Definition 2: A function $f: A \to B$ is even iff for every x, $f(-x) = f(x)$

Definition 2: A function f is periodic iff there exists a $k > 0$ such that for every x, $f(x + k) = f(x)$

Definition 3: A function f is non-decreasing iff for every x and for every y, $x \leq y \Rightarrow f(x) \leq f(y)$

Definition 4: A function f is increasing iff for every x and for every y, $x < y \Rightarrow f(x) < f(y)$

Definition 5: A function f is non-decreasing iff for every x and for every y, $x \leq y \Rightarrow f(x) \geq f(y)$

Definition 6: A function f is decreasing iff for every x and for every y, $x < y \Rightarrow f(x) > f(y)$

Definition 9: The number L is the limit of the function $f: A \to B$ at $a \in A$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

Definition 10: A function $f: A \to B$ is continuous at $a \in A$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

Definition 11: A function $f: A \to B$ is uniformly continuous on the set $C \subseteq A$ at $a \in A$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x \in C$, $y \in C$, and $|x - y| < \delta$.